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The Topological Correctness of PL-Approximations of Isomanifolds

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Abstract Isomanifolds are the generalization of isosurfaces to arbitrary dimension and codimension, i.e. manifolds defined as the zero set of some multivariate multivalued smooth function $f : \mathbb{R}^d \to \mathbb{R}^{d-n}$. A natural (and efficient) way to approximate an isomanifold is to consider its Piecewise-Linear (PL) approximation based on a triangulation $\mathcal{T}$ of the ambient space $\mathbb{R}^d$. In this paper, we give conditions under which the PL-approximation of an isomanifold is topologically equivalent to the isomanifold. The conditions are easy to satisfy in the sense that they can always be met by taking a sufficiently fine and thick triangulation $\mathcal{T}$. This contrasts with previous results on the triangulation of manifolds where, in arbitrary dimensions, delicate perturbations are needed to guarantee topological correctness, which leads to strong limitations in practice. We further give a bound on the Fréchet distance between the orig-

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inal isomanifold and its PL-approximation. Finally we show analogous results for the PL-approximation of an isomanifold with boundary.

1 Introduction

Isosurfacing

Given a surface represented in $\mathbb{R}^3$ as the zero set of a function $f : \mathbb{R}^3 \to \mathbb{R}$ the goal of isosurfacing is to find a piecewise linear (PL) approximation of the surface. This question naturally extends to higher dimensions and codimensions, in which case the generalized surface is called an isomanifold. Isosurfaces play a crucial role in medical imaging, computer graphics and geometry processing [46]. Higher dimensional isomanifolds are also of fundamental importance in many fields like statistics [22], dynamical systems [53], econometrics or mechanics [46].

Marching algorithms

The standard algorithmic solution to the isosurfacing problem is to use some marching algorithm. This approach was initiated by Lorensen and Cline, with their marching cube algorithm [41]. Many variants of the algorithm have been introduced, see for example [21, 32, 45, 54], and the overview [46]. The approach however is always the same: One first subdivides the ambient space into cubes (in which case the algorithm is called a marching cube algorithm), or simplices [1, 32, 45] (in which case the algorithm is called a marching tetrahedra algorithm). One starts with a cube or a simplex (cell) in which a part of the zero set of the function is contained, and finds a piecewise linear approximation of the zero set in that cell. One then propagates or marches to adjacent cells that also contain the zero set and approximates the zero set in that cell. This process can be continued until all cells that intersect the zero set have been visited.

For the marching cube algorithm [41] one also has to decide how to approximate the zero set inside a cube. As observed by Dürst, there is in general no canonical way how to do this due to *ambiguous configurations*, see [47] for an extensive discussion in the three dimensional setting. For the marching tetrahedra algorithm there is a canonical way to construct a piecewise linear approximation of the zero set, as we will discuss below. However the result of the algorithm is still not necessarily topologically correct.

Guarantees for Isosurfacing

For the marching simplex algorithm [32] in arbitrary dimensions, bounds have been given on the one-sided Hausdorff distance between the zero set of f and its PL approximation, and also on the difference between the gradient of f

and the gradient of the PL approximation. It can be proven that the result of the algorithm is a manifold under appropriate assumptions [2, 3].

An important requirement in the work of Allgower and Georg [3] is that the zero set avoids simplices that have dimension less than the codimension, see [3, Definition 12.2.2] and the text above [3, Theorem 15.4.1]. The idea to avoid these low dimensional simplices originates with Whitney [55], with whom Allgower and George [2, 3] were apparently unfamiliar. Very heavy perturbation schemes for the vertices of the ambient triangulation $\mathcal{T}$ are needed to ensure that the manifold stays sufficiently far from simplices in the ambient triangulation that have dimension less than the codimension of the manifold [17, 55]. Various techniques have been developed to compute such perturbations with guarantees. They typically consist in perturbing the position of the sample points or in assigning weights to the points. Complexity bounds are then obtained using volume arguments. See, for example [11, 13, 16, 23]. However, these techniques suffer from several drawbacks. The constants in the complexity depend exponentially on the ambient dimension. Moreover the analysis assumes that the probability of the simplices of dimension less than the codimension to intersect the manifold is zero, which is not true when dealing with finite precision. As a result, the actual implementations we are aware of fail to work well in practice except in very simple cases.

More complete correctness results have been achieved in three dimensions in the computational geometry community [12, 49] Boissonnat, Cohen-Steiner and Vegter [12] base their proof on a combination of Morse theory and simplicial collapses. Vegter and Plantinga's proof [49] is in its philosophy closely related to normal surface theory, see for example [52], but relies rather heavily on case analysis. The results of [12, 49] seem not extendable to higher dimensions.

Triangulating general manifolds (without boundary)

The approximation of a manifold that is the zero set of a function is an example of the more general question of how to triangulate a manifold. It is known that C^1 manifolds are triangulable, see for example [55], and algorithms have been proposed recently to triangulate smooth manifolds [9, 13, 14, 17]. However all known methods use intricate perturbation schemes to guarantee the correctness of the triangulation algorithms when the intrinsic dimension of the manifold exceeds 2. As for the case of isomanifolds, perturbation schemes work fine in theory but the constants are miserable and the methods do not work in practice in high dimensions.

Manifolds with boundary

In this paper, we also consider the piecewise linear approximation of manifolds with boundary (that are given as a zero set) and briefly mention the extension to stratifolds. Apart from some Delaunay based work on triangulations of stratifolds in three dimensions [29–31, 48, 50], we are not aware of similar results on manifolds with boundary. Significant effort also went in the detection of strata, in this case in arbitrary dimension, see for example [6, 7, 20].

Contribution

This paper contains three main results, the first two (Theorem 24 and Corollary 26) concern manifolds without boundary, the third (Theorem 47) manifolds with boundary.

Isomanifolds (without boundary) Let $f : \mathbb{R}^d \to \mathbb{R}^{d-n}$ be a smooth function and suppose that 0 is a regular value of f, meaning that at every point x such that $f(x) = 0$, the Jacobian of f is non-degenerate. Assume that $\mathcal{T}$ is a triangulation of $\mathbb{R}^d$. Define the function f_{PL} as the linear interpolation of the values of f at the vertices if restricted to a single simplex $\sigma \in \mathcal{T}$. Then,

Theorem 24 *The zero set of f_{PL} is a manifold that is ambient isotopic to the zero set of f, provided that the longest edge length D of $\mathcal{T}$ is sufficiently small (of the order of $1/d^2$).*

Corollary 26 (Bound on the Fréchet distance) *The Fréchet distance between f_{PL} and f is of the order of D^2, where D is the longest edge length of $\mathcal{T}$.*

We also give a variant of a result due to Allgower and George [2]:

Proposition 10 *The difference between the gradient of f and the gradient of its piecewise linear approximation is of order dD inside each simplex of $\mathcal{T}$.*

Isomanifolds with boundary Suppose that apart from f we are also given another function $f_\partial : \mathbb{R}^d \to \mathbb{R}$ and $f_{\partial,PL}$ is defined similarly to f_{PL}. Let us further assume that the zero set is regular in the following sense: The gradients of f^i span a $(d-n)$-dimensional space at each point of $f^{-1}(0)$ and the gradients of f^i and f_∂ span a $(d-n+1)$-dimensional space at each point of $\partial M = f^{-1}(0) \cap f_\partial^{-1}(0)$. Then,

Theorem 47 *The set $f^{-1}(0) \cap f_\partial^{-1}([0,\infty))$ is ambient isotopic to $f_{PL}^{-1}(0) \cap f_{\partial,PL}^{-1}([0,\infty))$, provided that the longest edge length D of $\mathcal{T}$ is sufficiently small (of the order of $1/d^2$).*

An important aspect of these results is that they hold under mild conditions: they simply ask for a sufficiently fine and thick triangulation $\mathcal{T}$. In contrast to previous results on the triangulation of manifolds, no perturbations are needed to guarantee topological correctness.

Our method provides guarantees on the Piecewise-Linear (PL) approximation of isomanifolds, regarding the topology, the Fréchet distance and the approximation of the gradients (the latter was already known to Allgower and Georg [2]). However, we stress that it does not give lower bounds on the quality of the linear pieces in the PL approximation. This is a clear difference with previous methods [14, 16, 17, 55] whose output is a thick triangulation. Although this is an appealing property, it complicates the analysis further and requires unpractical perturbation schemes. Such perturbation techniques could be added to our method to improve the simplex quality (to some limited extent). However, they are not required to make the algorithm work and to obtain the guarantees mentioned above.

The techniques used in this paper are also different from many of the standard tool and do not rely on Delaunay triangulations [24, 28], the closed ball property [4, 23, 36], Whitney's lemma [15] or collapses [5]. The current paper mainly relies on the non-smooth implicit function theorem [26] with some Morse theory.

Outline

The rest of this paper is subdivided as follows. In Section 2, we treat closed isomanifolds, i.e. compact manifolds without boundary. In Section 3, we treat isomanifolds with boundary. Extension to general isostratifolds is briefly discussed in Section 4. In the final section, we quantify the robustness of the method by studying how much the zero set of f changes if f is perturbed slightly in the C^1-sense [39].

This paper is closely related to another paper where the data structure needed to efficiently propagate along the manifold is presented [18]. Altogether, these two papers show that one can construct PL approximations of isomanifolds in space and time polynomial in the resolution $1/D$ of the ambient triangulation and in the dimension d of the ambient space.

2 Isomanifolds (without boundary)

Let $f : \mathbb{R}^d \to \mathbb{R}^{d-n}$ be a smooth (C^2 suffices) function and suppose that 0 is a regular value of f, meaning that at every point x such that $f(x) = 0$, the Jacobian of f is non-degenerate. Then the zero set of f is an n-dimensional manifold as a direct consequence of the implicit function theorem, see for example [33, Section 3.5]. We further assume that $f^{-1}(0)$ is compact. As in [2] we consider a triangulation $\mathcal{T}$ of $\mathbb{R}^d$. The function f_{PL} is the linear interpolation of the values of f at the vertices if restricted to a single simplex $\sigma \in \mathcal{T}$, i.e.

$$\forall x \in \sigma : \quad f_{PL}(x) = \sum_{v \in \sigma} \lambda_v(x) f(v),$$

where the λ_v are the barycentric coordinates of x with respect to the vertices of σ. For any function $g : \mathbb{R}^d \to \mathbb{R}^{d-n}$ we write g^i, with $i = 1, \ldots, d-n$, for the components of g.

We prove that under certain conditions there is an ambient isotopy from the zero set of f to the zero set of f_{PL}. The proof will be using the Piecewise-Linear (PL) map

$$F_{PL}(x, \tau) = (1 - \tau) f(x) + \tau f_{PL}(x), \tag{1}$$

which interpolates between f and f_{PL} and is based on the generalized implicit function theorem.

We are, by definition, only interested in $f^{-1}(0)$ and so can ignore points that are sufficiently far from this zero set. More precisely, we observe the following: If $f^i(x)$ is positive for all x in a geometric simplex σ then so is $f^i_{PL}(x)$ because $f^i_{PL}(x)$ is a convex combination of the (positive) values at the vertices. This in turn implies that $F^i_{PL}(x,\tau)$ is positive on $\sigma \times [0,1]$ as, for each τ, it is a convex combination of positive numbers. The same argument holds for negative values. So we see that

Remark 1 *Write $\mathcal{T}_0$ for the set of all $\sigma \in \mathcal{T}$, such that $(f^i)^{-1}(0) \cap \sigma \neq \emptyset$ for all i. Then for all τ, $\{x \mid F_{PL}(x,\tau) = 0\} \subset \mathcal{T}_0$.*

The results will be expressed using constants defined in terms of f and the ambient triangulation $\mathcal{T}$.

Definition 2 *We define*

$$\gamma_{\max} = \max_{x\in\mathcal{T}_0} (\max_i |\nabla f^i(x)|) \tag{2}$$

$$\lambda_{\min} = \min_{x\in\mathcal{T}_0} \lambda_{\min}(x), \tag{3}$$

$$\alpha_{\max} = \max_{x\in\mathcal{T}_0} \max_i \|Hes(f^i)(x)\|_2 \tag{4}$$

$$D : \textit{the longest edge length of a simplex in } \mathcal{T}_0 \tag{5}$$

$$T : \textit{the smallest thickness of a simplex in } \mathcal{T}_0. \tag{6}$$

Where

- $\nabla f^i = (\partial_j f_i)_j$ *denotes the gradient of component f^i, for $i \in [1, d-n]$,*
- $\mathrm{Gram}(\nabla f)$ *denotes the Gram matrix whose elements are $\nabla f^i \cdot \nabla f^j$ where $\cdot$ stands for the dot product.*
- $\lambda_{\min}(x)$ *denotes the smallest absolute value of the eigenvalues of* $\mathrm{Gram}(\nabla f(x))$,[1]
- $Hes(f) = (\partial_k \partial_l f_i)_{k,l}$ *denotes the Hessian matrix of second order derivatives,*
- $|\cdot|$ *denotes the Euclidean norm of a vector and $\|\cdot\|_2$ the operator 2-norm of a matrix.*[2]
- *The thickness is the ratio of the height (smallest altitude) over the longest edge length.*

We will assume that $\gamma_{\max}, \lambda_{\min}, \alpha_{\max}, D, T \in (0,\infty)$. The constant $\lambda_{\min}$ quantifies how close 0 is to not being a regular value of f. D is a measure of the size of the simplices of $\mathcal{T}$. We will call $\delta = 1/D$ the resolution of $\mathcal{T}$. The thickness is a quality measure of a simplex. A good choice for $\mathcal{T}$ is the Coxeter triangulation of type A_d, see [25, 27], or the related Freudenthal triangulations,

[1] Because a Gram matrix is a symmetric square matrix, its eigenvalues are well defined and real.

[2] The operator norm is defined as $\|A\|_p = \max_{x\in\mathbb{R}^n} \frac{|Ax|_p}{|x|_p}$, with $|\cdot|_p$ the p-norm on $\mathbb{R}^n$.

see [35, 37, 40, 53], which can be defined for different values of D while keeping T constant.

Our results hold for any dimensions d and n. We are especially interested in the case where the ambient dimension d is large. We thus consider d and D as the two main parameters and we think of all the other quantities as constants. For our bounds, will give both exact and asymptotic expressions. The asymptotic expressions are given to emphasize the dependency on the two most important parameters d and D, and hold for any d and D such that $dD < 1$ and any fixed positive $\gamma_{\max}, \lambda_{\min}, \alpha_{\max}$ and T. For convenience, exact expressions are gathered in Appendix A.

The result

We are going to construct an ambient isotopy based on (1). In fact, the map $\tau \mapsto \{x \mid F_{PL}(x,\tau) = 0\}$ gives an ambient isotopy between the zero set of $F_{PL}(x,0)$, which is identical to the smooth isosurface $f^{-1}(0)$, and the zero set of $F_{PL}(x,1)$, which is the PL approximation $f_{PL}^{-1}(0)$. The latter can be turned into a triangulation of the isosurface $f^{-1}(0)$ by triangulating the non simplicial cells using barycentric subdivision. We will also bound the Fréchet distance between $f^{-1}(0)$ and $f_{PL}^{-1}(0)$.

Proving the ambient isotopy consists of three technical steps. The first two consume most of the space in the proof below:

- *Local step.* Let $\sigma \in \mathcal{T}$. We first show that $\{(x,\tau) \mid F_{PL}(x,\tau) = 0\} \cap (\sigma \times [0,1])$ is a smooth manifold, under certain conditions (Corollary 12).
- *Global step.* We prove that $F_{PL}^{-1}(0)$ is a manifold, under certain conditions, using techniques from nonsmooth analysis (Corollary 23).

A crucial ingredient will be the implicit function theorem and its non-smooth extension. Along the way, we shall also see that $F_{PL}^{-1}(0)$ is never tangent to the $\tau = c$ planes, where c is a constant. The gradient of $(x,\tau), \mapsto \tau$ in $\mathbb{R}^d \times \mathbb{R}$ is $(0,1)$. Projecting this vector onto the tangent space of $F_{PL}^{-1}(0)$ gives the gradient of $(x,\tau), \mapsto \tau$ restricted to $F_{PL}^{-1}(0)$. Because of the non-tangency, this projection is non-zero. So the gradient field of the function $(x,\tau), \mapsto \tau$ restricted to $F_{PL}^{-1}(0)$, is piecewise smooth (because $F_{PL}^{-1}(0)$ is piecewise smooth) and never vanishes.

The third step is similar to a standard observation in Morse theory [42, 44], with the exception that we now consider piecewise-smooth instead of smooth vector fields. We refer to Milnor [42] for an excellent introduction, and to Lemma 2.4 and Theorem 3.1 in particular.

Lemma 3 (Gradient flow induced isotopies) *The flow of a non-vanishing piecewise-smooth gradient vector field of a function τ on a compact manifold generates a isotopy from $\tau = c_1$ to $\tau = c_2$, where c_1 and c_2 are constants.*

Proof This is a straightforward consequence of the existence and uniqueness of the solution to a differential equation. $\square$

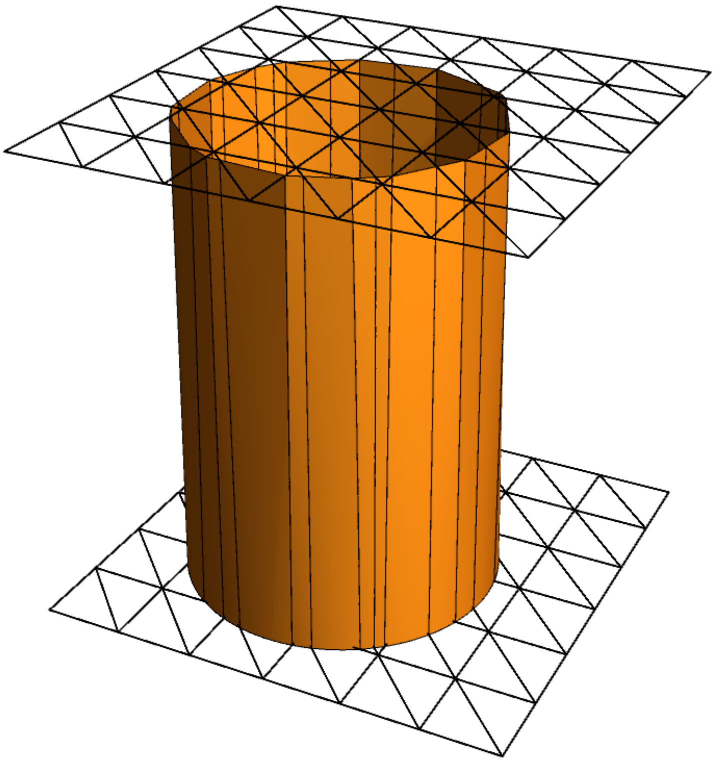

Fig. 1 A pictorial overview of the proof. The τ-direction goes upwards. Similarly to Morse theory, we find that $f_{PL}^{-1}(0)$ (top) and $f^{-1}(0)$ (bottom) are ambient isotopic if the function τ restricted to $F_{PL}^{-1}(0)$ does not encounter a Morse critical point.

Bounds on the gradient of τ on the manifold give a bound on the Fréchet distance, which is defined below:

Definition 4 (Fréchet distance for embedded manifolds) *Let $\mathcal{M}$ and $\mathcal{M}'$ be two homeomorphic, compact submanifolds of $\mathbb{R}^d$. Write $\mathcal{H}$ for the set of all homeomorphisms from $\mathcal{M}$ to $\mathcal{M}'$. The Fréchet distance between $\mathcal{M}$ and $\mathcal{M}'$ is*

$$d_F(\mathcal{M}, \mathcal{M}') = \inf_{h \in \mathcal{H}} \sup_{x \in \mathcal{M}} d(x, h(x)).$$

2.1 Preliminaries

The following elementary lemma will be useful.

Lemma 5

$$|f^i(x_1) - f^i(x_2)| \leq \gamma_{\max} |x_1 - x_2|. \tag{7}$$

$$|\nabla f^i(x_1) - \nabla f^i(x_2)| \leq d\alpha_{\max} \, |x_1 - x_2| \tag{8}$$

Proof The first statement follows from the fact that the supremum of the absolute value of a derivative of a function bounds the Lipschitz constant of the function. The second statement follows from standard bounds on matrix norm (see, for example, [38, Equation (2.3.11)]) together with (4). These bounds imply that $\sqrt{d}\alpha_{\max} \geq |\partial_k \partial_l f^i|$, for all k, l and i. Arguing as before, we deduce a bound on the Lipschitz constant of $\partial_l f^i$:

$$|\partial_l f^i(x_1) - \partial_l f^i(x_2)| \leq \sqrt{d}\alpha_{\max}|x_1 - x_2|.$$

The bound (8) now follows. □

2.1.1 The implicit function theorem

The main technical tool to prove the existence of the ambient isotopy from $f^{-1}(0)$ to $f_{PL}^{-1}(0)$ is the implicit function theorem which we recall now.

Theorem 6 (Smooth implicit function theorem) *Let $F : \mathbb{R}^{d+1} \to \mathbb{R}^{d-n}$ be a continuously differentiable function. Write $\mathbb{R}^{d+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{d-n}$ and denote the coordinates of $\mathbb{R}^{d+1}$ by (x, y) accordingly. Fix a point (a, b), with $F(a, b) = 0 \in \mathbb{R}^{d-n}$. If the Jacobian $J_{F,y}(a, b) = (\frac{\partial F^i}{\partial y^j}(a, b))_{i,j}$ is of maximal rank (or equivalently is invertible), then there exists an open set $U \subset \mathbb{R}^{n+1}$ containing a such that there exists a unique continuously differentiable function $g : U \to \mathbb{R}^{d-n}$ such that $g(a) = b$ and $F(x, g(x)) = 0$ for all $x \in U$.*

To prove the existence of the isotopy from $f^{-1}(0)$ to $f_{PL}^{-1}(0)$, we will apply the implicit function theorem (Theorem 6) to several functions g that are close to f and we will therefore need to prove that their Jacobians are of maximal rank. A matrix has maximal rank if and only if the Gram matrix of its columns has a non-zero determinant or, equivalently, non-zero eigenvalues. In our context, we will need lower bounds on the absolute values of the eigenvalues of the Gram matrices $\mathrm{Gram}(\nabla g)$, given the lower bound $\lambda_{\min}$ on the absolute values of the eigenvalues of $\mathrm{Gram}(\nabla f)$.

2.1.2 Eigenvalues and perturbations

We will follow the convention that the eigenvalues of the matrices we consider are sorted by increasing order of their absolute values, i.e. $|\lambda_i| \leq |\lambda_j|$ if $i \leq j$. We first recall Weyl's perturbation theorem that bounds the difference between the i-th eigenvalues of two symmetric matrices:

Lemma 7 (Weyl's bound, Corollary III.2.6 of [8]) *Let A and $\tilde{A} = A + E$ be two symmetric (or Hermitian) matrices and write λ_i and $\tilde{\lambda}_i$ for the eigenvalues of A and $\tilde{A}$ respectively. Then*

$$\max_i |\lambda_i - \tilde{\lambda}_i| \leq \|E\|_2,$$

where $\|\cdot\|_p$ denotes the p-norm.

We further note that $\|E\|_2 \leq \|E\|_F$ where $\|\cdot\|_F$ denotes the Frobenius norm, see [38, (2.3.7)]. By definition of the Frobenius norm, we have that $|E_{ij}| \leq e_{\max}$, for all $i, j \in [1, d-n]$, implies that $\|E\|_F \leq (d-n)e_{\max}$ if $d-n$ is the dimension of E. Hence, we have

Corollary 8 *Under the conditions of Lemma 7, and assumming* $\dim(E) = d-n$*, and* $|E_{ij}| \leq e_{\max}$ *we have*

$$\max_i |\lambda_i - \tilde{\lambda}_i| \leq (d-n)e_{\max}.$$

2.2 Estimates for a single simplex

In this section, we concentrate on a single simplex σ and write f_L for the linear function whose values on the vertices of σ coincide with f. In other words, f_L is the linear extension of the interpolation of f. Note that f_L coincides with f_{PL} within the geometric simplex σ (but not necessarily outside).

2.2.1 Estimates on the linear approximation f_L and its gradient

We need a simple estimate similar to Proposition 2.1 of Allgower and George [2].

Lemma 9 *Let $\sigma \subset \mathcal{T}_0$ and let f_L be as described above. Then, for all $x \in \sigma$,*

$$|f^i_L(x) - f^i(x)| \leq 2D^2\alpha_{\max}.$$

We included a proof for completeness.

Proof Let v_k be a vertex of σ. Taylor's theorem, see for example [33, Theorem 2.8.4], yields that

$$f^i(v_k) = f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k), \tag{9}$$

with

$$\begin{aligned} R(v_k) &= 2\sum_{j \neq l}(v_k - x)^j(v_k - x)^l \int_0^1 (1-t)^2 \partial_j\partial_l f^i(v_k - t(v_k - x))\mathrm{d}t \\ &\quad + 2\sum_j \frac{((v_k - x)^j)^2}{2} \int_0^1 (1-t)^2 \partial_j^2 f^i(v_k - t(v_k - x))\mathrm{d}t \\ &\leq 2|v_k - x|^2\alpha_{\max} && \text{(by (4) and Cauchy-Schwarz)} \\ &\leq 2D^2\alpha_{\max} && \text{(because } x \in \sigma) \end{aligned}$$

The function f_L at the point $x = \sum_k \lambda_k v_k$, where $\sum_k \lambda_k = 1$, is by construction

$$f^i_L(x) = \sum_k \lambda_k f^i(v_k)$$

$$
\begin{aligned}
&= \sum_k \lambda_k \left(f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k) \right) \qquad \text{(by (9))} \\
&= \sum_k \lambda_k f^i(x) + \sum_j \partial_j f^i(x)(\sum_k \lambda_k v_k - x)^j + \sum_k \lambda_k R(v_k) \\
&= f^i(x) + 0 + \sum_k \lambda_k R(v_k)
\end{aligned}
$$

Thanks to the bounds on $R(v_k)$ and Cauchy-Schwarz, one has

$$|f_L^i(x) - f^i(x)| \leq 2D^2 \alpha_{\max}$$

□

We will also be using an estimate similar to Proposition 2.2 of Allgower and George [2].

Proposition 10 *Let $\sigma \subset \mathcal{T}_0$ and let f_L be as described above. Then*

$$|\nabla f_L^i(x) - \nabla f^i(x)| = \sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2} \leq \frac{4dD\alpha_{\max}}{T},$$

for all x in the simplex σ.

We provide a proof for completeness.

Proof We again use that

$$f^i(v_k) = f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k), \tag{9}$$

with

$$|R(v_k)| \leq 2D^2 \alpha_{\max} \tag{10}$$

Subtracting $f^i(v_l)$ from $f^i(v_k)$ now yields

$$f^i(v_k) - f^i(v_l) = \sum_j \partial_j f^i(x)(v_k - v_l)^j + R(v_k) - R(v_l).$$

Because f_L is the linear interpolation of f, we have

$$f^i(v_k) - f^i(v_l) = \sum_j \partial_j f_L^i(x)(v_k - v_l),$$

and thus

$$\left| \sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))(v_k - v_l)^j \right| \leq |R(v_k) - R(v_l)| \leq 4D^2 \alpha_{\max}.$$

We now need a variant of a common refinement of two sets. Suppose we are given two finite sets $\{a_i \mid i = 0, \dots, i_{\max}\}, \{b_j \mid j = 0, \dots, j_{\max}\}$ of positive numbers such that $\sum a_i = \sum b_j$. Then there exists a set $\{c_k \mid k = 0, \dots, k_{\max}\}$ of positive integers $0 = k_0 \leq k_1 \leq \dots \leq k_{i_{\max}}$, $0 = k'_0 \leq k'_1 \leq \dots \leq k_{j_{\max}}$ such that

$$\sum_{k=k_i}^{k_{i+1}-1} c_k = a_i \qquad\qquad \sum_{k=k'_j}^{k'_{j+1}-1} c_k = b_j .$$

The proof is simple: you first pick $c_0 = \min(a_0, b_0)$ and then use induction.

Let now $u = \sum \mu_k v_k$ and $w = \sum \tilde{\mu}_k v_k$, with $\sum \mu_k = \sum \tilde{\mu}_k = 1$, that is $u, w \in \sigma$. We then split $u - w$ in positive and negative terms, where by a positive (negative) term we mean $(\mu_k - \tilde{\mu}_k) v_k$ such that $\mu_k - \tilde{\mu}_k$ is positive (negative). We note that the sum of the positive $(\mu_k - \tilde{\mu}_k)$s is less or equal to 1 and equals minus the sum of the negative $(\mu_k - \tilde{\mu}_k)$s. This means (by choosing the positive $(\mu_k - \tilde{\mu}_k)$s to equal the a_i and the negative ones b_j) that we can we write $u - v = \sum_k c_k (v_{m(k)} - v_{m(k')})$, with $\sum c_k \leq 1$ and $c_k > 0$.

We now see that

$$\begin{aligned}
&\left| \sum_j (\partial_j f^i_L(x) - \partial_j f^i(x))(u - w)^j \right| \\
&\quad = \left| \sum_{j,m} (\partial_j f^i_L(x) - \partial_j f^i(x)) c_m (v_{\tilde{k}(m)} - v_{\tilde{l}(m)})^j \right| \\
&\quad \leq \sum_m c_m \left| \sum_j (\partial_j f^i_L(x) - \partial_j f^i(x))(v_{\tilde{k}(m)} - v_{\tilde{l}(m)})^j \right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the triangle inequality and } c_m > 0) \\
&\quad \leq \sum_m 4 c_m D^2 \alpha_{\max} \qquad\qquad\qquad\qquad \text{(because } \textstyle\sum |c_m| \leq 1) \\
&\quad = 4 D^2 \alpha_{\max}. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (11)
\end{aligned}$$

Because the simplex σ contains a ball of radius the smallest altitude over d centred at its barycentre, that is TD/d with T the thickness, the vector $u - w$ can be chosen to be any vector of length less than tD/d. In particular we can choose

$$(u - w)^j = \frac{tD}{d} \frac{(\partial_j f^i_L(x) - \partial_j f^i(x))}{\sqrt{\sum_j (\partial_j f^i_L(x) - \partial_j f^i(x))^2}} .$$

Plugging this choice into (11) gives

$$\frac{tD}{d} \sqrt{\sum_j (\partial_j f^i_L(x) - \partial_j f^i(x))^2} \leq 4 D^2 \alpha_{\max}$$

So that

$$\sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2} \le \frac{4dD\alpha_{\max}}{T}.$$

□

We stress that the bound in Proposition 10 depends on the quality of the simplices in the ambient triangulation $\mathcal{T}$ but not on the shape of the cells of the PL approximation. This is fortunate since we know ambient triangulations of very good quality (e.g. Coxeter triangulations [25]) while we don't have control on the shapes of the cells of the PL approximation which depend on the way the isomanifold intersects $\mathcal{T}$.

2.2.2 Applying the implicit function theorem

Let σ be a simplex of $\mathcal{T}$ and let f_L be the linear approximation defined above. We now define a homotopy $F_L : \mathbb{R}^d \times [0,1] \to \mathbb{R}^{d-n}$:

$$F_L(x,\tau) = (1-\tau)f(x) + \tau f_L(x). \tag{12}$$

We intend to show that $F_L^{-1}(0)$ is a manifold in a neighbourhood of $\sigma \times [0,1]$. This follows from the Implicit function theorem (Theorem 6) provided that, for any point such that $F_L(x,\tau) = 0$ in this neighbourhood, the Jacobian is of maximal rank or, equivalently as recalled above, if and only if the Gram matrix of its columns has non-zero eigenvalues. The following lemma will provide lower bounds on the eigenvalues of this Gram matrix.

We denote by $\nabla_x F_L$ or simply ∇F_L the gradient of the restriction of F_L to the x variable and $\nabla_{x,\tau} F_L$ the gradient of F_L. Note that

$$\nabla_{x,\tau} F_L^i(x,\tau) = \begin{pmatrix} \nabla(f^i(x) + \tau(f_L^i(x) - f^i(x))) \\ f_L^i(x) - f^i(x) \end{pmatrix}. \tag{13}$$

For convenience, we will write $\nabla_{x,\tau} f(x) = \begin{pmatrix} \nabla f^i(x) \\ 0 \end{pmatrix}$.

Lemma 11 *Let* $\mathrm{G} = \mathrm{Gram}(\nabla f)$ *and* $\widehat{\mathrm{G}} = \mathrm{Gram}(\nabla_{x,\tau} F_L)$,[3] *and write* $\lambda_{\min}$ *and* $\widehat{\lambda}_{\min}$ *for the smallest absolute values of the eigenvalues of* G *and* $\widehat{\mathrm{G}}$ *respectively.*

$$|\widehat{\lambda}_{\min} - \lambda_{\min}| \le e_L \tag{14}$$

where $e_L = O(d^2 D)$*. The precise expression of* e_L *is given in* (16) *and* (15)*.*

[3] As a general rule, we put a $\widehat{\cdot}$ over quantities that are related to PL functions.

Proof Let, in addition to the notations of the lemma, $\widehat{\mathrm{G}}' = \mathrm{Gram}(\nabla_x F_L)$, $\lambda'_{\min}$ be the smallest absolute value of the eigenvalues of $\widehat{\mathrm{G}}'$, and write $\mathrm{G}_{i,j}$, $\widehat{\mathrm{G}}'_{i,j}$ and $\widehat{\mathrm{G}}_{i,j}$ for the entries of G, $\widehat{\mathrm{G}}'$ and $\widehat{\mathrm{G}}$ respectively. Proposition 10 yields that

$$
\begin{aligned}
&|\widehat{\mathrm{G}}'_{i,j}(x) - \mathrm{G}_{i,j}(x)| \\
&\quad = |\nabla(f^i(x) + \tau(f_L(x)^i - f^i(x))) \cdot \nabla(f^j(x) + \tau(f_L(x)^j - f^j(x))) \\
&\qquad - \nabla(f^i(x)) \cdot \nabla(f^j(x))| \\
&\quad = |\nabla(\tau(f_L(x)^i - f^i(x))) \cdot \nabla(f^j(x)) \\
&\qquad + \nabla(\tau(f_L(x)^j - f^j(x))) \cdot \nabla(f^i(x)) \\
&\qquad + \nabla(\tau(f_L(x)^i - f^i(x))) \cdot \nabla(\tau(f_L(x)^j - f^j(x)))| \\
&\quad \leq 2\gamma_{\max} \frac{4dD\alpha_{\max}}{T} + \left(\frac{4dD\alpha_{\max}}{T}\right)^2 \stackrel{\text{def}}{=} e'_L.
\end{aligned}
\tag{15}
$$

The addition of the τ component gives a small extra contribution.

$$
\begin{aligned}
|\widehat{\mathrm{G}}_{i,j}(x) - \mathrm{G}_{i,j}(x)| &= |\nabla_{x,\tau} F^i_L(x) \cdot \nabla_{x,\tau} F^j_L(x) - \nabla(f^i(x)) \cdot \nabla(f^j(x))| \\
&= |\nabla(f^i(x) + \tau(f^i_L(x) - f^i(x))) \cdot \nabla(f^j(x) + \tau(f^j_L(x) - f^j(x))) \\
&+ (f^i_L(x) - f^i(x))(f^j_L(x) - f^j(x)) - \nabla(f^i(x)) \cdot \nabla(f^j(x))| \\
&\leq e'_L + (2D^2\alpha_{\max})^2. \qquad \text{(by Lemma 9)}
\end{aligned}
$$

Applying Corollary 8, we obtain

$$
|\widehat{\lambda}_{\min} - \lambda_{\min}| \geq (d-n)(e'_L + (2D^2\alpha_{\max})^2) \stackrel{\text{def}}{=} e_L. \tag{16}
$$

From (15) we see that $e'_L = O(dD)$, hence $e_L = O(d^2D)$. □

The following corollary follows directly from the previous lemma and the discussion before the lemma.

Corollary 12 ($F_L^{-1}(0)$ is a manifold in a neighbourhood of $\sigma \times [0,1]$) *Under the Regularity condition*

$$
\lambda_{\min} > e_L \tag{17}
$$

the implicit function theorem applies to $F_L(x,\tau)$ *inside* $\sigma \times [0,1]$. *(In fact it applies to an open neighbourhood of this set). It follows that* $\{(x,\tau) \mid F_L(x,\tau) = 0\} \cap (\sigma \times [0,1])$ *is a smooth manifold.*

2.2.3 Transversality with regard to the τ-direction

We now prove that inside each $\sigma \times [0,1]$ the gradient of τ on $F_L = 0$ is smooth and does not vanish.

We need the following straightforward lemma. We include a proof for completeness.

Lemma 13 *Now suppose that $A = (v_i)^t(v_i)$ is a Gram matrix, where (v_i) denotes the matrix whose column are the vectors v_i, that is $A_{ij} = v_i \cdot v_j$. Similarly to before denote by $\lambda_{\min}(A)$ the smallest absolute value of an eigenvalue of the Gram matrix A. We have that $\sqrt{\lambda_{\min}(A)} \leq |v_k|$, for all k.*

Proof We see that

$$\begin{aligned}
\lambda_{\min} &= \min_{|u|=1} |u^t A u| \\
&= \min_{|u|=1} |u^t (v_i)^t (v_i) u| \\
&= \min_{|u|=1} |((v_i)u)^t ((v_i)u)| \\
&= \min_{|u|=1} |((v_i)u)|^2 \\
&\leq \min_{u=e_j} |((v_i)u)|^2 \\
&= \min_j |v_j|^2 \\
&\leq |v_k|^2
\end{aligned}$$

□

We also need to bound the angle of the vectors $\nabla_{x,\tau}(F_L^i)$ and the x plane, that is $\mathbb{R}^d \subset \mathbb{R}^{d+1}$. We recall the definition. If $v \in \mathbb{R}^{d+1}$ is a vector and $\varXi = \mathbb{R}^d \subset \mathbb{R}^{d+1}$ is the space spanned by the d basis vectors corresponding to the x-directions, the angle between v and $\varXi$ is

$$\angle(v, \varXi) = \inf_{w \in \varXi} \angle(v, w).$$

Lemma 14 *Let $\varXi$ be as above. We have*

$$\tan \angle(\nabla_{x,\tau} F_L^i, \varXi) \leq \theta \stackrel{\text{def}}{=} \frac{2D^2 \alpha_{\max}}{\sqrt{\lambda_{\min}} - \frac{4dD\alpha_{\max}}{T}}. \tag{18}$$

In particular, the manifold $F_L^{-1}(0)$ inside $\sigma \times [0,1]$ is never tangent to the $\tau = c$ planes, where c is a constant, provided that the following Transversality condition holds

$$\lambda_{\min} > \left(\frac{4dD\alpha_{\max}}{T} \right)^2. \tag{19}$$

Proof By (13), the absolute value of the τ-component of $\nabla_{x,\tau} F^i_L$ is $|f_L(x)^i - f^i(x)|$, which is upper bounded by $2D^2\alpha_{\max}$ (Lemma 9). On the other hand, the norm of the x-component of $\nabla_{x,\tau} F^i_L$ is lower bounded by

$$\sqrt{\lambda_{\min}} - \frac{4dD\alpha_{\max}}{T},$$

as a consequence of Proposition 10 and Lemma 13. The result now follows since $\tan \angle(\nabla_{x,\tau} F^i_L, \Xi)$ is the ratio between the absolute value of the τ-component and the norm of the x-component of $\nabla_{x,\tau} F^i_L$. $\square$

From Corollary 12 and Lemma 14, we immediately deduce

Corollary 15 *Under the Regularity and Transversality conditions* (17) *and* (19)*, which both holds for $D = O(1/d^2)$, the gradient of τ on $F_L^{-1}(0)$ is smooth and does not vanish inside $\sigma \times [0,1]$ for any $\sigma \in \mathcal{T}_0$.*

2.3 Global result

2.3.1 The non-smooth implicit function theorem

For the global result, we need to recall some definitions and results from non-smooth analysis. We refer to [26] for an extensive introduction.

Definition 16 (Generalized Jacobian, Definition 2.6.1 of [26]) *Let $F : \mathbb{R}^{d+1} \to \mathbb{R}^{d-n}$, where F is assumed to be just Lipschitz. The generalized Jacobian of F at x_0, denoted by $J_F(x_0)$, is the convex hull of all $(d-n) \times (d+1)$-matrices B obtained as the limit of a sequence of the form $J_F(x_i)$, where $x_i \to x_0$ and F is differentiable at x_i.*

Following [26, page 253] we also define:

Definition 17 *The generalized Jacobian $J_F(x_0)$ is said to be of maximal rank provided every matrix in $J_F(x_0)$ is of maximal rank.*

Write $\mathbb{R}^{d+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{d-n}$ and denote the coordinates of $\mathbb{R}^{d+1}$ by (x, y) accordingly. Fix a point (a, b), with $F(a, b) = 0 \in \mathbb{R}^{d-n}$. We now write:

Notation 18 ([26, page 256]) *$J_F(x_0, y_0)|_y$ is the set of all $(n+1) \times (n+1)$-matrices M such that, for some $(n+1) \times (d-n)$-matrix N, the $(n+1) \times (d+1)$-matrix $[N, M]$ belongs to $J_F(x_0, y_0)$.*

With these definitions and notations, we now have:

Theorem 19 (Generalized implicit function theorem [26, page 256]) *Suppose that $J_F(a, b)|_y$ is of maximal rank. Then there exists an open set $U \subset \mathbb{R}^{n+1}$ containing a such that there exists a Lipschitz function $g : U \to \mathbb{R}^{d-n}$, such that $g(a) = b$ and $F(x, g(x)) = 0$ for all $x \in U$.*

2.3.2 Applying the non-smooth implicit function theorem

We recall the definition of F_{PL}:

$$F_{PL}(x,\tau) = (1-\tau)f(x) + \tau f_{PL}(x). \tag{1}$$

Further recall that the closed star of a vertex v in a simplical complex is the closure of all simplices in the complex that contain v. We will also be using the following remark often.

Remark 20 *Let v be a vertex in $\mathcal{T}$, $x_1, x_2 \in star(v)$, then*

$$|x_1 - x_2| \leq 2D. \tag{20}$$

We now have

Lemma 21 *Let v be a vertex in $\mathcal{T}$, $x_1, x_2 \in star(v)$, and $\tau_1, \tau_2 \in [0,1]$, such that $\nabla_{x,\tau} F^i_{PL}(x_1,\tau_1)$ and $\nabla_{x,\tau} F^i_{PL}(x_2,\tau_2)$ are well defined, then*

$$|\nabla_{x,\tau} F^i_{PL}(x_1,\tau_1) - \nabla_{x,\tau} F^i_{PL}(x_2,\tau_2)| \leq g_{PL} \stackrel{\text{def}}{=} 2dD\alpha_{\max} + \frac{8dD\alpha_{\max}}{T} + 4D^2\alpha_{\max}. \tag{21}$$

Proof Because

$$\nabla_{x,\tau} F^i_{PL}(x_1,\tau_1) = \begin{pmatrix} \nabla(f^i(x_1) + \tau_1(f^i_{PL}(x_1) - f^i(x_1))) \\ f^i_{PL}(x_1) - f^i(x_1) \end{pmatrix},$$

we get

$$\begin{aligned} &|\nabla_{x,\tau} F^i_{PL}(x_1,\tau_1) - \nabla_{x,\tau} F^i_{PL}(x_2,\tau_2)| \\ &\quad \leq |\nabla f^i(x_1) - \nabla f^i(x_2)| \\ &\qquad + |\nabla f^i_{PL}(x_1) - \nabla f^i(x_1)| + |\nabla f^i_{PL}(x_2) - \nabla f^i(x_2)| \qquad (\tau_i \in [0,1]) \\ &\qquad + |f^i_{PL}(x_1) - f^i(x_1)| + |f^i_{PL}(x_2) - f^i(x_2)| \\ &\quad \leq 2dD\alpha_{\max} + \frac{8dD\alpha_{\max}}{T} + 4D^2\alpha_{\max} \\ &\qquad\qquad \text{(by (8) and (20), Proposition 10 and Lemma 9)} \end{aligned}$$

□

We generalize Lemma 11 as follows.

Lemma 22 *Let v be a vertex in $\mathcal{T}$, $x_1, \ldots, x_m \in star(v)$, and $\tau_1, \ldots, \tau_m \in [0,1]$. We assume that $\nabla_{x,\tau} F^i_{PL}(x_k,\tau_k)$ is well defined for $k = 1, \ldots, m$, define $\widehat{\mathbf{G}} = \mathrm{Gram}(\sum_{k=1}^m \mu_k \nabla_{x,\tau} F_{PL}(x_k,\tau_k))$, and let $\widehat{\mathbf{\Lambda}}_{\min}$ be the smallest modulus of the eigenvalues of $\widehat{\mathbf{G}}$. Then,*

$$|\widehat{\mathbf{\Lambda}}_{\min} - \widehat{\lambda}_{\min}| \leq e_{PL},$$

where $e_{PL} = O(d^2 D)$ and is precisely defined in (23).

Proof Let $\widehat{\mathrm{G}} = \mathrm{Gram}(\nabla_{x,\tau} F_L(x_0, y_0))$ and let $\widehat{\lambda}_{\min}$ be the smallest modulus of the eigenvalues of $\widehat{\mathrm{G}}$. We claim that the elements of the two matrices $\widehat{\mathbf{G}}$ and $\widehat{\mathrm{G}}$ are pairwise close. Specifically, using the identity $A \cdot B - C \cdot D = A \cdot (B - D) + (A - C) \cdot D$:

$$
\begin{aligned}
&|\widehat{\mathbf{G}}_{i,j} - \widehat{\mathrm{G}}_{i,j}| \\
&\leq \Bigg| \sum_k \mu_k \nabla_{x,\tau} F^i_{PL}(x_k, \tau_k) \cdot \sum_k \mu_k \nabla_{x,\tau} F^j_{PL}(x_k, \tau_k) \\
&\quad - \sum_k \mu_k \nabla_{x,\tau} F^i_{PL}(x_0, \tau_0) \cdot \sum_k \mu_k \nabla_{x,\tau} F^j_{PL}(x_0, \tau_0) \Bigg| \\
&= \Bigg| \left(\sum_k \mu_k \nabla_{x,\tau} (F^i_{PL}(x_k, \tau_k) \right) \cdot \left(\sum_k \mu_k \left(\nabla_{x,\tau} F^j_{PL}(x_k, \tau_k) - \nabla_{x,\tau} F^j_{PL}(x_0, \tau_0) \right) \right) \\
&\quad + \left(\sum_k \mu_k \left(\nabla_{x,\tau} F^i_{PL}(x_k, \tau_k) - \nabla_{x,\tau} F^i_{PL}(x_0, \tau_0) \right) \right) \cdot \left(\sum_k \mu_k \nabla_{x,\tau} (F^j_{PL}(x_0, \tau_0) \right) \Bigg| \\
&\leq g_{PL} \cdot \sum_k \mu_k \left(\left| \nabla_{x,\tau} F^i_{PL}(x_k, \tau_k) \right| + \left| \nabla_{x,\tau} F^j_{PL}(x_0, \tau_0) \right| \right) \Bigg|, \\
&\qquad\qquad\qquad\qquad \text{(by Cauchy-Schwarz and the triangle inequality)}
\end{aligned}
$$

where g_{PL} is given in Lemma 21. It remains to bound $|\nabla_{x,\tau} F^i_{PL}(x_k, \tau_k)|$:

$$
\begin{aligned}
|\nabla_{x,\tau} F^i_{PL}(x_k, \tau_k)| &\leq |\nabla_x \left(f^i(x_k) + \tau (f^i_{PL}(x_k) - f^i(x_k)) \right)| + |f^i_{PL}(x_k) - f^i(x_k)))| \\
&\leq \gamma_{\max} + \frac{4dD\alpha_{\max}}{T} + 2D^2 \alpha_{\max},
\end{aligned}
\tag{22}
$$

where we used Lemma 9 and Proposition 10. We conclude that

$$
|\widehat{\mathbf{G}}_{i,j} - \widehat{\mathrm{G}}_{i,j}| \leq g_{PL} (\gamma_{\max} + \frac{4dD\alpha_{\max}}{T} + 2D^2 \alpha_{\max}).
$$

Applying Corollary 8, we get

$$
|\widehat{\mathbf{\Lambda}}_{\min} - \widehat{\lambda}_{\min}| \leq (d - n) g_{PL} (\gamma_{\max} + \frac{4dD\alpha_{\max}}{T} + 2D^2 \alpha_{\max}) \stackrel{\text{def}}{=} e_{PL} \tag{23}
$$

which completes the proof of the lemma. □

From the previous lemmas we immediately have that,

Corollary 23 ($\{(x, \tau) \mid F_{PL}(x, \tau) = 0\}$ is a manifold) *Under the regularity condition*

$$
\lambda_{\min} > e_{PL}, \tag{24}
$$

where $e_{PL} = O(d^2 D)$ is precisely defined in (23)*, the generalized implicit function theorem, Theorem 19, applies to $F_{PL}(x, \tau) = 0$. In particular, $\{(x, \tau) \mid F_{PL}(x, \tau) = 0\}$ is a manifold.*

The second technical step of the proof is now completed. The third step follows from an application of Lemma 3. The fact that $F_L(x,\tau) = 0$ is a Piecewise-Smooth manifold and transversality, as proven in Lemma 14, gives that the gradient of τ is a Piecewise-Smooth vector field whose flow we can integrate to give an ambient isotopy from the zero set of f to that of f_{PL}.

We summarize in a theorem:

Theorem 24 *If the regularity condition* (24) *and the transversality condition* (19) *hold, the zero set of f_{PL} is a manifold isotopic to the zero set of f. Note that both conditions hold when $D = O(1/d^2)$.*

2.3.3 Fréchet distance

To bound the Fréchet distance, denoted by d_F, between the zero sets of $f(x)$ and f_{PL}, it suffices to bound the angle that the gradient of τ (as restricted to $F_L(x,\tau) = 0$) makes with the τ-direction (in $\mathbb{R}^{d+1}$). We write e_τ for the unit vector in the τ direction (again in $\mathbb{R}^{d+1}$).

For this, we will use the angle bound of Lemma 14, together with some estimates that are similar in spirit to those in [10, Lemma C.13].

Lemma 25 *For any $w \in \operatorname{span}(\nabla_{x,\tau} F_L^i)$, we have*

$$\cos\angle(w, e_\tau) \leq \frac{\sin(\theta)(\gamma_{\max} + \frac{4dD\alpha_{\max}}{T})}{\sqrt{\lambda_{\min} - e_{PL}}} = O(D^2)$$

where $e_{PL} = O(d^2 D)$ is defined in (23) *and $\theta = O(D^2)$ is defined in* (18).

Proof Write $v_i = \nabla_{x,\tau} F_L^i$ for $i \in [1, d-n]$, and $w = \mu_1 v^1 + \cdots + \mu_{d-n} v^{d-n}$ with $\mu_1, \ldots, \mu_{d-n} \in \mathbb{R}$. We have $|w|^2 = \sum_{i,j} \mu_i \mu_j v^i \cdot v^j$, and, by definition,

$$\cos\angle(w, e_\tau) = \frac{\sum_i \mu_i v^i \cdot e_\tau}{|w|} \quad \text{and} \quad |w|^2 = \sum_{i,j} \mu_i \mu_j v^i \cdot v^j \geq \widehat{\boldsymbol{\Lambda}}_{\min} |\mu|^2, \tag{25}$$

where $\widehat{\boldsymbol{\Lambda}}_{\min}$ is defined in Lemma 22. Proposition 10 and $|\nabla(f^i)| \leq \gamma_{\max}$ give

$$|v^i| \leq \gamma_{\max} + \frac{4dD\alpha_{\max}}{T}. \tag{26}$$

Lemma 14 states that

$$\angle(\nabla_{x,\tau} F_L^i, \Xi) \leq \theta = \arctan \frac{2D^2 \alpha_{\max}}{\sqrt{\lambda_{\min}} - \frac{4dD\alpha_{\max}}{T}}.$$

By definition, Ξ is the space orthogonal to e_τ (with e_τ aligned with the τ direction), so that

$$\cos(\angle \nabla_{x,\tau} F_L^i, e_\tau) = \sin\angle(\nabla_{x,\tau} F_L^i, \Xi). \tag{27}$$

Hence, by definition of the cosine and (26), we see

$$|v^i \cdot e_\tau| \leq \sin(\theta) \left(\gamma_{\max} + \frac{4dD\alpha_{\max}}{T} \right)$$

Using (25) and Cauchy-Schwarz, we then obtain

$$\cos \angle(w, e_\tau) \leq \frac{|\mu| \sin(\theta)(\gamma_{\max} + \frac{4dD\alpha_{\max}}{T})}{\sqrt{\widehat{\mathbf{\Lambda}}_{\min} |\mu|^2}} = \frac{\sin(\theta)(\gamma_{\max} + \frac{4dD\alpha_{\max}}{T})}{\sqrt{\widehat{\mathbf{\Lambda}}_{\min}}}$$

The result now follows thanks to Lemma 22. □

Let F_{PL_0} be the restriction of F_{PL} to $F_{PL}^{-1}(0, \tau)$. Moreover, and $\nabla_\tau F_{PL_0}$ be the gradient of τ restricted to $F_{PL}^{-1}(0)$, whenever it exists. We want to bound the angle of $\nabla_\tau F_{PL_0}$ and the τ-direction. Because the isotopy is given by the gradient flow and we have a bound on the norm of the gradient, the Fréchet distance is bounded. Specifically, the bound is equal to the norm of the gradient since the time we follow the flow is 1.

There is one subtlety. Because the manifold is only Piecewise-Smooth, we need to take into account the points where $\nabla_\tau F_{PL_0}$ is not uniquely defined. Because, for each simplex σ, F_L extends to a neighbourhood of $\sigma \times [0, 1]$, there exists a limit of $\nabla_\tau F_{PL_0}(x_i, \tau_i)$ for any sequence (x_i, τ_i) that lies in $\mathrm{int}(\sigma) \times [0, 1]$, where int denotes the interior. This means that, if we bound $\nabla_\tau F_{PL_0}$ for each simplex, we also bound its limits, where the limits are as just described.

Corollary 26 (Bound on the Fréchet distance) *Suppose that the conditions of Theorem 24 are satisfied. Then,*

$$d_F(f^{-1}(0), f_{PL}^{-1}(0)) \leq d_{PL}$$

where $d_{PL} = O(D^2)$ *is defined in* (28).

Proof Let, as before, $\Xi = \mathbb{R}^d \subset \mathbb{R}^{d+1}$ be the space spanned by the d basis vectors corresponding to the x-directions. Lemma 25 gives, for $w \in \mathrm{span}_i(\nabla_{x,\tau}(F^i))$,

$$\cos \angle(w, e_\tau) \leq \frac{\sin(\theta)(\gamma_{\max} + \frac{4dD\alpha_{\max}}{T})}{\sqrt{\lambda_{\min} - e_{PL}}}.$$

Since the tangent space to $F_L = 0$ is normal to $\mathrm{span}_i(\nabla_{x,\tau}(F_L^i))$, the same bound holds for $\sin \angle(\nabla_\tau F_{PL_0}, e_\tau)$. This means that, as $\tau \in [0, 1]$, the distance between the begin and the end points of the gradient flow, and thus the Fréchet distance, is bounded by $\tan \angle(\nabla_\tau F_{PL_0}, e_\tau)$, that is

$$d_F(f^{-1}(0), f_{PL}^{-1}(0)) \leq \tan \arcsin \frac{\sin(\theta)(\gamma_{\max} + \frac{4dD\alpha_{\max}}{T})}{\sqrt{\lambda_{\min} - e_{PL}}} \stackrel{\text{def}}{=} d_{PL} \tag{28}$$

The asymptotic dependence follows because $\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$, and $\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$. □

3 Isomanifolds with boundary

We will now consider isomanifolds with boundary. By this, we mean that on top of the function $f : \mathbb{R}^d \to \mathbb{R}^{d-n}$, we will have another function $f_\partial : \mathbb{R}^d \to \mathbb{R}$ and the set we consider is $M = f^{-1}(0) \cap f_\partial^{-1}([0,\infty))$. This is a manifold with boundary if the gradients of f^i span a $(d-n)$-dimensional space at each point of $f^{-1}(0)$ and the gradients of f^i and f_∂ span a $(d-n+1)$-dimensional space at each point of $\partial M = f^{-1}(0) \cap f_\partial^{-1}(0)$, as a consequence of the submersion theorem.

We will again write f_{PL} for the PL interpolation of f. Similarly, we write $f_{\partial,PL}$ for the PL interpolation of f_∂.

We prove that, under certain conditions, there is an isotopy from $f^{-1}(0) \cap f_\partial^{-1}([0,\infty))$ to $f_{PL}^{-1}(0) \cap f_{\partial,PL}^{-1}([0,\infty))$. The conditions are very similar to the conditions we have before but, of course, we need to include bounds on the gradient of $f_{\partial,PL}$.

Overview of the proof

We will again construct an isotopy but in this case it will consist of two steps.

- In the **first step**, we isotope the part of $f^{-1}(0)$ that is far from $f_\partial^{-1}(0)$ to its piecewise linear approximation, while leaving the part of $f^{-1}(0)$ that is close to $f_\partial^{-1}(0)$ smooth. We will denote the result by $M_1 = (F_{PL,1}(\cdot,1))^{-1}(0)$.
- In the **second step**, we consider a (small) tubular neighbourhood around $f_\partial^{-1}(0)$ as restricted to M_1 by looking at all $f_\partial^{-1}(\epsilon)$ for $|\epsilon|$ sufficiently small.[4] We then isotope $M_1 \cap f_\partial^{-1}(\epsilon)$ to its piecewise linear approximation. Again, the isotopy is chosen in such a way that, for ϵ relatively large, it leaves $M_1 \cap f_\partial^{-1}(\epsilon)$ invariant (for the points such that M_1 is already Piecewise-Linear). This gives an isotopy of a tubular neighbourhood of $M_1 \cap f_\partial^{-1}(0)$ to its Piecewise-Linear approximation.

We will first partition the manifold in two parts using a smooth bump function $\phi : \mathbb{R} \to [0,1]$, defined so that $\phi(y) = 0$ in a neighbourhood of zero and $\phi(y) = 1$ if $|y| > y_0$, for some $y_0 > 0$. Such bump functions can be easily constructed, see for example [39, Section 2.2]. We will be using the function $\phi\left(\sum_i (f^i)^2 + f_\partial^2\right)$ often. In fact, because it is used so often, it will be convenient to introduce the following shorthand

$$|f_B|^2 = \sum_i (f^i)^2 + f_\partial^2 \tag{29}$$

The first step will be using the zero set of the following function:

$$F_{PL,1}(x,\tau) = \left(1 - \tau\phi\left(|f_B|^2\right)\right) f(x) + \tau\phi\left(|f_B|^2\right) f_{PL}(x), \tag{30}$$

on which we will apply the same gradient flow argument as before.

[4] We stress that ϵ may be negative.

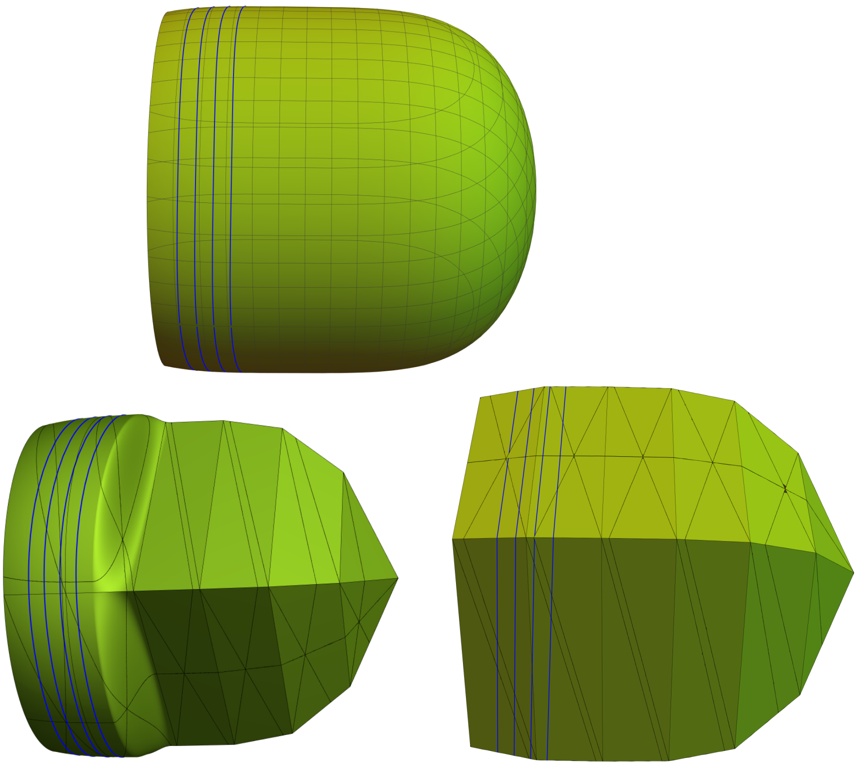

Fig. 2 Top: we see the original isosurface with $f_\partial^{-1}(-1/10)$, $f_\partial^{-1}(0)$, $f_\partial^{-1}(1/10)$, and $f_\partial^{-1}(2/10)$ indicated in blue. Bottom left: we see that at the end of Step 1 the neighbourhood of the boundary is intact, while the rest has been isotoped to a Piecewise-Linear approximation. Bottom right: we have also isotoped the neighbourhood of the boundary to a Piecewise-Linear approximation by isotoping $f_\partial^{-1}(\epsilon)$, to its Piecewise-Linear approximation for all sufficiently small ϵ.

The resulting set M_1 is the same zero set of f_{PL} as before if we stay sufficiently far away from ∂M and the isotopy leaves the manifold invariant close to ∂M. In particular, $\partial M_1 = \partial M$.

In the second step, we define an isotopy that will act only on a small neighbourhood of ∂M. Consider the sets $B_1(\epsilon) = M_1 \cap f_\partial^{-1}(\epsilon)$ and, for each of them, define the function

$$\begin{aligned} F_{PL,2,\epsilon} :& B_1(\epsilon) \times [0,1] \to \mathbb{R}^{d-n+1} : \\ F_{PL,2,\epsilon}(x,\tau) &= \left(1 - \tau\psi\left(|f_B|^2\right)\right) \left(F_{PL,1}(x,1), f_\partial(x) - \epsilon\right) \\ &\quad + \tau\psi\left(|f_B|^2\right) \left(f_{PL}(x), f_{\partial,PL}(x) - \epsilon\right), \end{aligned} \tag{31}$$

where $\psi : \mathbb{R} \to [0,1]$ is now a smooth bump function that is 1 in a sufficiently large neighbourhood of zero (somewhat larger than y_0) and zero outside some compact set. Using the result for isomanifolds (with some modifications), we can prove that each individual set $B_1(\epsilon)$ is isotopic to $f_{PL}^{-1}(0) \cap f_{\partial,PL}^{-1}(\epsilon)$ for small ϵ while, for sufficiently large ϵ, it leaves the set invariant.

3.1 Step 1

The proof closely follows the proof for the case without boundary in Section 2. The main technical difficulty will be to provide bounds that serve as the counterparts to Lemma 21 for both steps in the proof. To be able to do so, we first need to discuss bounds on the bump functions ϕ and ψ.

3.1.1 Bump functions

Following [39, Section 2.2] we write,

$$\zeta_1(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

For $0 < y_1 < y_2$ we write $\zeta_2(x) = \zeta_1(x-y_1)\zeta_1(y_2-x)$. Then we define $\phi_l : \mathbb{R} \to [0,1]$ by $\phi_l(x) = \int_x^{y_2} \zeta_2(x')\mathrm{d}x' \Big/ \int_{y_1}^{y_2} \zeta_2(x')\mathrm{d}x'$. Finally define $\phi_b : \mathbb{R} \to [0,1]$ by $\phi_b(x) = \phi_l(|x|)$, and let $\phi(x) = 1 - \phi_b(x)$.

Lemma 27 *We have $\phi_b(x) \in [0,1]$ and, writing $2y_1 = y_2 = y_0$,*

$$\partial_x(\phi_l(x)) \leq 2\frac{e^{\frac{4}{3(y_2-y_1)}}}{y_2-y_1} = 4\frac{e^{\frac{2}{3y_0}}}{y_0} = \gamma_\phi. \tag{32}$$

Proof As mentioned, by construction $\phi_b(x) \in [0,1]$. Because $\partial_x\beta\left(\frac{y_1+y_2}{2}\right) = 0$ and this is the only zero of the derivative in the open interval (y_1, y_2), we see that

$$\beta(x) \leq \beta\left(\frac{y_1+y_2}{2}\right) = e^{\frac{4}{y_1-y_2}}.$$

Hence, because $\beta(x) \geq 0$,

$$\int_{y_1}^{y_2} \beta(x) \leq (y_2-y_1)e^{\frac{4}{y_1-y_2}}$$

Because $\beta(x)$ is monotone on $[y_1, \frac{y_1+y_2}{2}]$ we also have

$$\int_{y_1}^{y_2} \beta(x) \geq \frac{y_2-y_1}{2}\beta\left(\frac{3}{4}y_1 + \frac{1}{4}y_2\right) = \frac{y_2-y_1}{2}e^{\frac{16}{3(y_1-y_2)}}$$

We now have

$$
\begin{aligned}
\partial_x(\phi_l(x)) &= \partial_x \left(\int_x^{y_2} \beta(x')\mathrm{d}x' \Big/ \int_{y_1}^{y_2} \beta(x')\mathrm{d}x' \right) \\
&= \beta(x) / \int_{y_1}^{y_2} \beta(x')\mathrm{d}x' \\
&\leq \frac{e^{\frac{4}{y_1-y_2}}}{\frac{y_2-y_1}{2} e^{\frac{16}{3(y_1-y_2)}}} \\
&= 2\frac{e^{\frac{4}{y_1-y_2}-\frac{16}{3(y_1-y_2)}}}{y_2-y_1} \\
&= 2\frac{e^{\frac{4}{3(y_2-y_1)}}}{y_2-y_1}
\end{aligned}
$$

□

3.1.2 Inside a single simplex

Similarly to Corollary 12, we need a condition that ensures that the zero set of $F^i_{PL,1}(x,\tau)$ restricted to $\sigma \times [0,1]$is a smooth manifold. In fact, similarly to (12), we define

$$
\begin{aligned}
F^i_{L,1}(x,\tau) &= \left(1 - \tau\phi\left(|f_B|^2\right)\right) f^i(x) + \tau\phi\left(|f_B|^2\right) f^i_L(x) \\
&= f^i(x) + \tau\phi\left(|f_B|^2\right)(f^i_L(x) - f^i(x)),
\end{aligned}
$$

where ϕ is as defined above. Observe that $F^i_{L,1}(x,\tau)$ can be extended to a neighbourhood of $\sigma \times [0,1]$.

Remark 28 *For the constants, it is better if y_0 can be chosen as large as possible, but we need y_1 to be quite a bit larger than y_0. In turn, we cannot choose y_1 arbitrarily large because this would mean that the gradient field $\nabla f_\partial|_{f^{-1}(0)}$ (seen as restricted on $f^{-1}(0)$) would never vanish. The latter is in general impossible thanks to the hairy ball theorem [19].*

We introduce the following definition that complements Definition 2:

Definition 29

$$
\Gamma^B_{\max} = \max_{x\in\mathcal{T}_0} \left|\nabla\left(|f_B|^2\right)\right| = 2\max_{x\in\mathcal{T}_0}\left|\sum_l f^l \nabla f^l + f_\partial \nabla f_\partial\right| \tag{33}
$$

$$
\hat{\lambda}^{B_1}_{\min} = \min_{x\in\mathcal{T}_0} \lambda_{\min}(\widehat{\mathrm{G}}^{B_1})(x). \tag{34}
$$

where $\widehat{\mathrm{G}}^{B_1} = \mathrm{Gram}(\nabla_{x,\tau} F_{L,1})$ and $\lambda_{\min}(A)$ denotes, as before, the smallest absolute value of the eigenvalues of matrix A.

We have then the analog of Lemma 11:

Lemma 30 *We have,*

$$|\hat{\lambda}^{B_1}_{\min} - \lambda_{\min}| \leq e^{B_1}_L$$

where $e^{B_1}_L = O(d^2 D)$ *is precisely defined in* (38) *and* (35).

Proof We start with an estimate on the individual $\nabla_{x,\tau} F^i_{L,1}(x,\tau)$. As noted before, we write for convenience $\nabla_{x,\tau} f(x) = \begin{pmatrix} \nabla f^i(x) \\ 0 \end{pmatrix}$.

$$\begin{aligned}
&\left|\nabla_{x,\tau} f(x) - \nabla_{x,\tau} F^i_{L,1}(x,\tau)\right| \\
&= \left|\nabla_{x,\tau} f(x) - \nabla_{x,\tau}\left(f(x) + \tau\phi\left(|f_B|^2\right)\left(f^i_L(x) - f^i(x)\right)\right)\right| \\
&= \left|-\nabla_{x,\tau}\left(\tau\phi\left(|f_B|^2\right)\left(f^i_L(x) - f^i(x)\right)\right)\right| \\
&= \left|\begin{pmatrix} -\tau\nabla\left(\phi\left(|f_B|^2\right)\right)\left(f^i_L(x) - f^i(x)\right) \\ \phi\left(|f_B|^2\right)\left(f^i_L(x) - f^i(x)\right)\end{pmatrix} + \begin{pmatrix} -\tau\phi\left(|f_B|^2\right)\nabla(f^i_L(x) - f^i(x)) \\ 0 \end{pmatrix}\right| \\
&\leq \left|\tau\nabla\left(\phi\left(|f_B|^2\right)\right)\left(f^i_L(x) - f^i(x)\right)\right| + \left|\tau\phi\left(|f_B|^2\right)\nabla(f^i_L(x) - f^i(x))\right| \\
&\quad + \left|\phi\left(|f_B|^2\right)\left(f^i_L(x) - f^i(x)\right)\right| \qquad \text{(by the triangle inequality)} \\
&\leq \mathit{\Gamma}^B_{\max}\gamma_\phi |f^i_L(x) - f^i(x)| + \left|\phi\left(|f_B|^2\right)\nabla(f^i_L(x) - f^i(x))\right| + |f^i_L(x) - f^i(x)| \\
&\qquad \text{(because } \tau \leq 1, \text{ (33), (32), } \phi \in [0,1]) \\
&\leq \mathit{\Gamma}^B_{\max}\gamma_\phi |f^i_L(x) - f^i(x)| + \left|\nabla(f^i_L(x) - f^i(x))\right| + |f^i_L(x) - f^i(x)| \\
&\qquad \text{(because } \phi \in [0,1]) \\
&\leq 2D^2\alpha_{\max} + 2\mathit{\Gamma}^B_{\max}\gamma_\phi D^2\alpha_{\max} + \frac{4dD\alpha_{\max}}{T} \stackrel{\text{def}}{=} 2D^2\alpha_{\max} + e^{B_1}_t \qquad (35) \\
&\qquad \text{(by Lemma 9 and Proposition 10)}
\end{aligned}$$

We now write $\mathrm{G}_{i,j}$ and $\widehat{\mathrm{G}}^{B_1}_{i,j}$ for the elements (i,j) of G and $\widehat{\mathrm{G}}^{B_1}$ respectively. Expanding yields

$$\begin{aligned}
\widehat{\mathrm{G}}^{B_1}_{i,j} &= \left(\nabla_{x,\tau} F^i_{L,1}(x,\tau) - \nabla_{x,\tau} f^i(x) + \nabla_{x,\tau} f^i(x)\right) \\
&\quad \cdot \left(\nabla_{x,\tau} F^j_{L,1}(x,\tau) - \nabla_{x,\tau} f^j(x)\right) \\
&= \mathrm{G}_{i,j} + \left(\nabla_{x,\tau} F^i_{L,1}(x,\tau) - \nabla_{x,\tau} f^j(x)\right) \cdot \nabla_{x,\tau} f^i(x) \\
&\quad + \nabla_{x,\tau} f^j(x) \cdot \left(\nabla_{x,\tau} F^j_{L,1}(x,\tau) - \nabla_{x,\tau} f^j(x)\right) \\
&\quad + \left(\nabla_{x,\tau} F^i_{L,1}(x,\tau) - \nabla_{x,\tau} f^j(x)\right) \cdot \left(\nabla_{x,\tau} F^j_{L,1}(x,\tau) - \nabla_{x,\tau} f^j(x)\right) \qquad (36)
\end{aligned}$$

We now see by Cauchy-Schwarz, the triangle inequality and Equation (35) that

$$|\widehat{\mathrm{G}}^{B_1}_{i,j} - \mathrm{G}_{i,j}| \leq 2\gamma_{\max}\left(2D^2\alpha_{\max} + e^{B_1}_t\right) + \left(2D^2\alpha_{\max} + e^{B_1}_t\right)^2 \qquad (37)$$

Corollary 8 now yields $\hat{\lambda}^{B_1}_{\min} > \lambda_{\min} - e^{B_1}_L$ where

$$e^{B_1}_L \stackrel{\text{def}}{=} (d-n)\left(2\gamma_{\max}\left(2D^2\alpha_{\max} + e^{B_1}_t\right) + \left(2D^2\alpha_{\max} + e^{B_1}_t\right)^2\right) \qquad (38)$$

□

and $e_t^{B_1}$ is defined in (35). The following corollary is then the analog of Corollary 12:

Corollary 31 ($F_{L,1}^{-1}(0)$ is a manifold) *If the regularity condition*

$$\lambda_{\min} > e_L^{B_1}, \tag{39}$$

holds, then $F_{L,1}^{-1}(0)$ is a smooth manifold inside an ϵ neighbourhood of $\sigma \times [0,1]$.

3.1.3 Transversality with regard to the τ-direction

We note that, similarly to Lemma 14, we have

Lemma 32 *Using the notation of Lemma 14, we have*

$$\tan \angle(\nabla_{x,\tau}(F_{L,1}), \Xi) \leq \frac{2D^2 \alpha_{\max}}{\sqrt{\lambda_{\min}} - e_t^{B_1}}$$

Proof The proof is identical to the proof of Lemma 14 with the replacement of $\frac{4dD\alpha_{\max}}{T}$ in the denominator by $e_t^{B_1}$ The latter constant is a consequence of (35). □

Now, similarly to Corollary 15, we find that

Corollary 33 (Transversality with respect to τ for Step 1) *Assume that both the Regularity condition* (39) *and the transversality condition*

$$\lambda_{\min} > e_t^{B_1} \tag{40}$$

hold. Then, inside each $\sigma \times [0,1]$, the gradient of τ on $F_{L,1}^{-1}(0)$ is smooth and does not vanish. Both conditions are satisfied if $D = O(1/d^2)$.

3.1.4 Global result

We now have to prove that $F_{PL,1}^{-1}(0)$ is a manifold. For this we again employ the generalized implicit function theorem. But first of all, we need the following bound, which is similar to Lemma 21.

Note that $\nabla_{x,\tau} F^i(x_0, \tau_0)$ is well defined as soon as x_0 lies in the interior of a d-simplex in $\mathcal{T}$.

Lemma 34 *Assuming that the gradients are well defined, we have*

$$|\nabla_{x,\tau} F^i_{PL,1}(x_1, \tau_1) - \nabla_{x,\tau} F^i_{PL,1}(x_2, \tau_2)| \leq g_{PL}^{B_1},$$

where $g_{PL}^{B_1} = \mathcal{O}(dD)$ is precisely defined in (41).

Proof By expansion we see that

$$
\begin{aligned}
&|\nabla_{x,\tau}F^i_{PL,1}(x_1,\tau_1) - \nabla_{x,\tau}F^i_{PL,1}(x_2,\tau_2)| \\
&\quad = \big|\nabla_{x,\tau}F^i_{PL,1}(x_1,\tau_1) - \nabla_{x,\tau}f^i(x_1) + \nabla_{x,\tau}f^i(x_1) \\
&\qquad - \nabla^i_{x,\tau f}(x_2) + \nabla_{x,\tau}f^i(x_2) - \nabla_{x,\tau}F^i_{PL,1}(x_2,\tau_2)\big| \\
&\quad \le \big|\nabla_{x,\tau}F^i_{PL,1}(x_1,\tau_1) - \nabla_{x,\tau}f^i(x_1)\big| \\
&\qquad + \big|\nabla_{x,\tau}f^i(x_1) - \nabla_{x,\tau}f^i(x_2)\big| \\
&\qquad + \big|\nabla_{x,\tau}f^i(x_2) - \nabla_{x,\tau}F^i_{PL,1}(x_2,\tau_2)\big| && \text{(by the triangle inequality)} \\
&\quad \le 2d\alpha_{\max}D + 4D^2\alpha_{\max} + 2e^{B_1}_t && \text{(by (20), (8), and (35) twice)} \\
&\quad \stackrel{\text{def}}{=} g^{B_1}_{PL} && (41)
\end{aligned}
$$

This completes the proof. □

Suppose $x_0, x_1, \ldots, x_m \in \mathrm{star}(v)$, $\tau_0, \ldots, \tau_m \in [0,1]$ and that $\nabla_{x,\tau}F^i_{PL,1}(x_i,\tau_i)$ is well defined for all i. Further assume that $\mu_1, \ldots, \mu_m$ are positive weights such that $\mu_1 + \cdots + \mu_m = 1$. We write

$$
\widehat{\mathbf{G}}^{B_1} = \mathrm{Gram}\left(\sum_{k=1}^{m} \mu_k \nabla_{x,\tau}F_{PL,1}(x_k,\tau_k)\right),
$$

and $\widehat{\mathbf{\Lambda}}^{B_1}_{\min}$ for the smallest modulus of the eigenvalues of $\widehat{\mathbf{G}}^{B_1}$.

Lemma 35 *We have*

$$
|\widehat{\mathbf{\Lambda}}^{B_1}_{\min} - \lambda_{\min}| \le e^{B_1}_{PL}
$$

with $e^{B_1}_{PL} = O(d^2D)$ is precisely defined in (44).

Proof The proof is more or less the same as the proof of Lemma 22, but with more complicated bounds. We assume that $x_0 \in \mathrm{star}(v)$ and $\tau_0 \in [0,1]$ are such that $\nabla_{x,\tau}F^i(x_0,\tau_0)$ is well defined (i.e. x_0 lies in the interior of a d-simplex of $\mathcal{T}$). Lemma 30 gives that

$$
\widehat{\mathbf{\Lambda}}^{B_1}_{\min} > \lambda_{\min} - e^{B_1}_L
$$

Using $\nabla(f^i) \le \gamma_{\max}$ and (35), we note that

$$
\big|\nabla_{x,\tau}F^i_{PL,1}(x_0,\tau_0)\big| \le \gamma_{\max} + 2D^2\alpha_{\max} + e^{B_1}_t \tag{42}
$$

We want to use Weyl's bound to determine a bound on the smallest absolute value of the eigenvalues of $\widehat{\mathbf{G}}^{B_1}$. Writing $\widehat{\mathrm{G}}^{B_1}_{i,j}$ and $\widehat{\mathbf{G}}^{B_1}_{i,j}$ for element (i,j) of matrices $\widehat{\mathrm{G}}^{B_1}(x_0,\tau_0)$ and $\widehat{\mathbf{G}}^{B_1}$ respectively, we show that $\widehat{\mathbf{G}}^{B_1}_{i,j}$ and $\widehat{\mathrm{G}}^{B_1}_{i,j}(x_0,\tau_0)$ are pairwise close (compare to (37)).

$$
|\widehat{\mathbf{G}}^{B_1}_{i,j} - \widehat{\mathrm{G}}^{B_1}_{i,j}(x_0,\tau_0)|
$$

$$
\begin{aligned}
&= \Bigg| \left(\sum_{k=1}^{m} \mu_k \, \nabla_{x,\tau} F^i_{PL,1}(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^{m} \mu_k \, \nabla_{x,\tau} F^j_{PL,1}(x_k, \tau_k) \right) \\
&\qquad - \nabla_{x,\tau} F^i_{PL,1}(x_0, \tau_0) \cdot \nabla_{x,\tau} F^i_{PL,1}(x_0, \tau_0) \Bigg| \\
&\leq \left| \nabla_{x,\tau} F^i_{PL,1}(x_0, \tau_0) - \sum_{k=1}^{m} \mu_k \nabla_{x,\tau} F^i_{PL,1}(x_k, \tau_k) \right| \\
&\quad \cdot \left| \nabla_{x,\tau} F^j_{PL,1}(x_0, \tau_0) \right| \\
&\quad + \left| \nabla_{x,\tau} F^j_{PL,1}(x_0, \tau_0) - \sum_{k=1}^{m} \mu_k \nabla_{x,\tau} F^j_{PL,1}(x_k, \tau_k) \right| \\
&\quad \cdot \left| \nabla_{x,\tau} F^i_{PL,1}(x_0, \tau_0) \right| \\
&\quad + \left| \nabla_{x,\tau} F^i_{PL,1}(x_0, \tau_0) - \sum_{k=1}^{m} \mu_k \nabla_{x,\tau} F^i_{PL,1}(x_k, \tau_k) \right| \\
&\qquad \cdot \left| \nabla_{x,\tau} F^j_{PL,1}(x_0, \tau_0) - \sum_{k=1}^{m} \mu_k \nabla_{x,\tau} F^j_{PL,1}(x_k, \tau_k) \right| \\
&\leq 2 g^{B_1}_{PL} \cdot \left(\gamma_{\max} + 2D^2 \alpha_{\max} + e^{B_1}_t + \left(g^{B_1}_{PL} \right)^2 \right) \qquad (43) \\
&\qquad\qquad\qquad \text{(by Lemma 34 and (42))}
\end{aligned}
$$

Using the result of Lemma 30 and invoking Corollary 8 once more gives

$$
|\widehat{\mathbf{\Lambda}}^{B_1}_{\min} - \lambda_{\min}| \leq e^{B_1}_{PL}
$$

with

$$
e^{B_1}_{PL} \stackrel{\text{def}}{=} e^{B_1}_L - (d-n) \left(2 g^{B_1}_{PL} \cdot \left(\gamma_{\max} + 2D^2 \alpha_{\max} + e^{B_1}_t \right) + \left(g^{B_1}_{PL} \right)^2 \right) \tag{44}
$$

□

Lemma 35 immediately yields that

Corollary 36 ($\boldsymbol{F^{-1}_{PL,1}(0)}$ is a manifold) *If the Regularity condition*

$$
\lambda_{\min} > e^{B_1}_{PL} \tag{45}
$$

holds, then the generalized implicit function theorem, Theorem 19, applies to $F_{PL,1}(x, \tau) = 0$. *In particular* $F^{-1}_{PL,1}(0)$ *is a manifold.*

We stress again that, inside the set $\{x | \phi \left(\sum_i (f^i)^2(x) + f^2_\partial(x) \right) = 1\}$, the zero set of $F_{PL,1}(x, 1)$ coincides with the zero set of $f_{PL}(x)$.

3.2 Step 2

Before we can proceed we have to specify the bump function ψ. We suppose that

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{101}{100} y_0 \\ 0 & \text{if } |x| \geq 2y_0. \end{cases}$$

In particular,

Definition 37 *We pick $\psi(x) = \phi_b(x)$, with the choice $y_1 = \frac{101}{100} y_0$ and $y_2 = 2y_0$.*

Remark 38 *We stress that the choice of y_1 and y_2 for the function ψ is different from the choice we made for ϕ, in the first step of the proof.*

First, we stress that the zero set of $F_{PL,2,\epsilon}(x,1)$ coincides with the zero set of $(f_{PL}(x), f_{\partial,PL}(x) - \epsilon)$, provided that $\psi(\sum_i f_i(x)^2 + f_\partial(x)^2) = 1$.

Secondly, we now claim the following:

Lemma 39 *The zero set of $F_{PL,2,\epsilon}(x,1)$ is a subset of the zero set of $f_{PL}(x)$, for each ϵ.*

Proof We focus on the first $d - n$ coordinates of (31). We see that

$$\begin{aligned}
&\left(1 - \psi\left(|f_B|^2\right)\right) F_{PL,1}(x,1) + \psi\left(|f_B|^2\right) f_{PL}(x) \\
&= \left(1 - \psi\left(|f_B|^2\right)\right) \left(\left(1 - \phi\left(|f_B|^2\right)\right) f(x) + \phi\left(|f_B|^2\right) f_{PL}(x)\right) + \psi\left(|f_B|^2\right) f_{PL}(x) \\
&= \left(1 - \psi\left(|f_B|^2\right)\right) \left(1 - \phi\left(|f_B|^2\right)\right) f(x) + \left(\left(1 - \psi\left(|f_B|^2\right)\right) \phi\left(|f_B|^2\right) + \psi\left(|f_B|^2\right)\right) f_{PL}(x) \\
&= \left(\left(1 - \psi\left(|f_B|^2\right)\right) \phi\left(|f_B|^2\right) + \psi\left(|f_B|^2\right)\right) f_{PL}(x),
\end{aligned} \tag{46}$$

where we used that

$$\begin{aligned}
1 - \psi\left(|f_B|^2\right) &= 0, \quad \text{if} \quad |f_B|^2 \leq \frac{101}{100} y_0, \\
\left(1 - \phi\left(|f_B|^2\right)\right) &= 0, \quad \text{if} \quad |f_B|^2 \geq y_0.
\end{aligned}$$

We can further rewrite (46),

$$\begin{aligned}
&\left(\left(1 - \psi\left(|f_B|^2\right)\right) \cdot \left(\phi\left(|f_B|^2\right) - 1 + 1\right) + \psi\left(|f_B|^2\right)\right) f_{PL}(x) \\
&= \left(\left(1 - \psi\left(|f_B|^2\right)\right) \left(\phi\left(|f_B|^2\right) - 1\right) + 1\right) f_{PL}(x) \\
&= f_{PL}(x),
\end{aligned}$$

where we used the same argument as before. □

The technical result that remains to be proven is the counterpart of Theorem 24 for $F_{PL,2,\epsilon}(x,\tau)$ and for each sufficiently small ϵ. To be precise, it suffices for $\epsilon \leq 2y_0$. We remark that it is likely that this bound on ϵ can be improved.

We again follow the same path to prove this result. That is: we first concentrate on a single simplex and prove that inside that simplex the zero set of $F_{PL,2,\epsilon}$ is a smooth manifold on which the gradient of τ as restricted to the manifold does not vanish. We then prove that the zero set of $F_{PL,2,\epsilon}$ is globally a manifold.

3.2.1 Assumptions and notations

Because we are now faced with both $f(x)$ and $f_\partial(x)$ we need to introduce a bound on how far the gradients of all the entrees of these functions are from being collinear. We write

$$f_B(x) = (f(x), f_\partial(x)). \tag{47}$$

Before we were only interested in the set $\mathcal{T}_0$. Similarly here, we sometimes concentrate on a neighbourhood of the zero set of both f_∂ and f. Therefore we write $\mathcal{T}_B$ for all $\sigma \in \mathcal{T}$ such that $(\sum_l (f^l)^2 + (f_\partial)^2)^{-1}([-2y_0, 2y_0]) \cap \sigma \neq \emptyset$.

We also write $\mathrm{G}^B = \mathrm{Gram}(\nabla f_B)$ and $\lambda^B_{\min}$ for the minimal absolute value of the eigenvalues of G^B, where the minimization is over all simplices in the set $\mathcal{T}_B \cap \mathcal{T}_0$. The restriction to the set $\mathcal{T}_B \cap \mathcal{T}_0$ is important, because if the minimization would be just over $\mathcal{T}_0$, G^B would generically be 0 as a consequence of the hairy ball theorem.

We note that by taking gradients the ϵ constant drops from the expression, so that the properties we now define are independent of ϵ. For the lengths of the gradients of f_B we define,

$$\gamma^B_{\max} = \max_{x \in \mathcal{T}_0} \max_i |\nabla(f^i_B)|, \tag{48}$$

for all $1 \leq i \leq d - n + 1$. Similarly to $\alpha_{\max}$, we define $\alpha^B_{\max}$ as the bound on the operator 2-norm of all Hessians of f_B, that is

$$\alpha^B_{\max} = \max_{x \in \mathcal{T}_0} \max_i \|\mathrm{Hes}(f^i_B)\|_2 = \max_{x \in \mathcal{T}_0} \max_i \|(\partial_k \partial_l f^i_B)_{k,l}\|_2. \tag{49}$$

We stress that that $\alpha_{\max} \leq \alpha^B_{\max}$.

We use the same notation for the ambient triangulation $\mathcal{T}$, the lower bound on the thickness of the simplices T and upper bound on the longest edge length D. We also need to introduce a bound on the differential of the bump function ψ. Similarly to (32), we define

$$\gamma_\psi = 2 \frac{e^{\frac{4}{3(y_2 - y_1)}}}{y_2 - y_1} = 2 \frac{e^{\frac{4}{3(2y_0 - \frac{101}{100} y_0)}}}{2y_0 - \frac{101}{100} y_0} = \frac{200}{99} \frac{e^{\frac{400}{297 y_0}}}{y_0}, \tag{50}$$

because we picked $y_1 = \frac{101}{100} y_0$ and $y_2 = 2y_0$, for ψ, see Definition 37 and Remark 38.

3.2.2 Inside a single simplex

Similarly to Lemma 30, we now give a condition that ensures that the zero set of $F_{PL,2,\epsilon}(x, \tau)$ is smooth inside $\sigma \times [0, 1]$. In fact, similarly to (12), we define

$$\begin{aligned} F_{L,2,\epsilon}(x, \tau) = &\left(1 - \tau\psi\left(|f_B|^2\right)\right)\left(F_{L,1}(x, 1), f_\partial(x) - \epsilon\right) \\ &+ \tau\psi\left(|f_B|^2\right)\left(f_L(x), f_{\partial,L}(x) - \epsilon\right), \end{aligned}$$

which can be extended to a neighbourhood of $\sigma \times [0,1]$.

We also write $\widehat{\mathbf{G}}^{B_2} = \mathrm{Gram}(\nabla_{x,\tau} F_{PL,2,\epsilon}$ and $\hat{\lambda}^{B_2}_{\min}$ for the smallest absolute value of the eigenvalues of $\widehat{\mathbf{G}}^{B_2}$.

Lemma 40 *For all* ϵ

$$|\hat{\lambda}^{B_2}_{\min} - \lambda^{B}_{\min}| \leq e_L^{B2},$$

where $e_L^{B2} = O(d^2 D)$ *is precisely defined in* (60).

Proof We start with an estimate on the individual $\nabla_{x,\tau} F^i_{L,2,\epsilon}(x,\tau)$. We will write $(v,w)^i$ for the i-th coordinate of the composed vector (v,w). We now see that

$$\begin{aligned}
&\left|\nabla_{x,\tau} f_B^i(x) - \nabla_{x,\tau} F^i_{L,2,\epsilon}(x,\tau)\right| \\
&= \Big|\nabla_{x,\tau} f_B^i(x) \\
&\quad - \nabla_{x,\tau}\left(\left(1-\tau\psi\left(|f_B|^2\right)\right)\left(F_{L,1}(x,1), f_\partial(x)-\epsilon\right)^i\right. \\
&\quad \left. +\tau\psi\left(|f_B|^2\right)\left(f_L(x), f_{\partial,L}(x)-\epsilon\right)^i\right)\Big| \\
&= \Big|\nabla_{x,\tau} - \nabla_{x,\tau}\left(\left(1-\tau\psi\left(|f_B|^2\right)\right)\right. \\
&\quad \cdot \left(\left(1-\phi\left(|f_B|^2\right)\right)f(x) + \phi\left(|f_B|^2\right)f_L(x), f_\partial(x)-\epsilon\right)^i \\
&&& \text{(by definition of } F_{L,1}) \\
&\quad \left. +\tau\psi\left(|f_B|^2\right)\left(f_L(x), f_{\partial,L}(x)-\epsilon\right)^i\right)\Big| \\
&= \Big|\nabla_{x,\tau} f_B^i(x) - \nabla_{x,\tau}\left(\left(1-\tau\psi\left(|f_B|^2\right)\right)\right. \\
&\quad \cdot \left(f(x) + \phi\left(|f_B|^2\right)\left(f_L(x)-f(x)\right), f_\partial(x)-\epsilon\right)^i \\
&\quad \left. +\tau\psi\left(|f_B|^2\right)\left(f_L(x), f_{\partial,L}(x)-\epsilon\right)^i\right)\Big| \\
&= \Bigg|\begin{pmatrix}\nabla f_B^i(x) \\ 0\end{pmatrix} \\
&\quad - \nabla_{x,\tau}\Big(\left(f(x) + \phi\left(|f_B|^2\right)\left(f_L(x)-f(x)\right), f_\partial(x)-\epsilon\right)^i \\
&\quad + \tau\psi\left(|f_B|^2\right) \\
&\quad \cdot \left(f_L(x) - f(x) - \phi\left(|f_B|^2\right)\left(f_L(x)-f(x)\right), f_{\partial,L}(x) - f_\partial(x)\right)^i\Big)\Bigg| \\
&= \Big|-\nabla_{x,\tau}\Big(\left(\phi\left(|f_B|^2\right)\left(f_L(x)-f(x)\right), 0\right)^i &&& \text{(by definition of } f_B^i(x)) \\
&\quad + \tau\psi\left(|f_B|^2\right) \\
&\quad \cdot \left(\left(1-\phi\left(|f_B|^2\right)\right)\left(f_L(x)-f(x)\right), f_{\partial,L}(x)-f_\partial(x)\right)^i\Big)\Big| \\
&= \Bigg|\begin{pmatrix}-\nabla\left(\phi\left(|f_B|^2\right)\left(f_L(x)-f(x)\right), 0\right)^i \\ 0\end{pmatrix} \\
&\quad + \begin{pmatrix}\tau\nabla\left(\psi\left(|f_B|^2\right)\left(\left(1-\phi\left(|f_B|^2\right)\right)\left(f_L(x)-f(x)\right), f_{\partial,L}(x)-f_\partial(x)\right)^i\right) \\ \psi\left(|f_B|^2\right)\left(\left(1-\phi\left(|f_B|^2\right)\right)\left(f_L(x)-f(x)\right), f_{\partial,L}(x)-f_\partial(x)\right)^i\end{pmatrix}\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \max_j \left|\nabla \left(\phi\left(|f_B|^2\right)\left(f_L^j(x) - f^j(x)\right)\right)\right| && \text{(by the triangle inequality)}\\
&\quad + \left|\nabla\left(\psi\left(|f_B|^2\right)\right.\right. \\
&\quad \cdot \left.\left.\left(\left(1-\phi\left(|f_B|^2\right)\right)\left(f_L(x) - f(x)\right), f_{\partial,L}(x) - f_\partial(x)\right)^i\right)\right| && \text{(because } \tau \in [0,1])\\
&\quad + \left|\left(\left(f_L(x) - f(x)\right), f_{\partial,L}(x) - f_\partial(x)\right)^i\right| \\
& && \text{(because } \phi(y), \psi(y) \in [0,1], \text{ for all } y)\\
&\leq \max_j \left|\nabla\phi\left(|f_B|^2\right)\right| |f_L^j(x) - f^j(x)| \\
&\quad + \max_j \left|\phi\left(|f_B|^2\right)\right| |\nabla(f_L^j(x) - f^j(x))| \\
& && \text{(by the Leibniz rule, Cauchy-Schwarz, and the triangle inequality)}\\
&\quad + \left|\nabla\left(\psi\left(|f_B|^2\right)\right)\right| \\
&\quad \cdot \left|\left(\left(1-\phi\left(|f_B|^2\right)\right)\left(f_L(x) - f(x)\right), f_{\partial,L}(x) - f_\partial(x)\right)^i\right| \\
&\quad + \left|\psi\left(|f_B|^2\right)\right| \\
&\quad \cdot \left|\nabla\left(\left(\left(1-\phi\left(|f_B|^2\right)\right)\left(f_L(x) - f(x)\right), f_{\partial,L}(x) - f_\partial(x)\right)^i\right)\right| \\
& && \text{(by the Leibniz rule, and the triangle inequality)}\\
&\quad + 2D^2\alpha^B_{\max} && \text{(by Lemma 9)}\\
&\leq \max_j \gamma_\phi \mathit{\Gamma}^B_{\max} |f_L^j(x) - f^j(x)| + \max_j |\nabla(f_L^j(x) - f^j(x))| \\
& && \text{(by Lemmas 27, (32), (33), and since } \phi(y) \in [0,1])\\
&\quad + \gamma_\psi \mathit{\Gamma}^B_{\max} \left|\left(\left(1-\phi\left(|f_B|^2\right)\right)\left(f_L(x) - f(x)\right), f_{\partial,L}(x) - f_\partial(x)\right)^i\right| \\
& && \text{(by Lemma 27, (33), (50))}\\
&\quad + \left|\nabla\left(\left(\left(1-\phi\left(|f_B|^2\right)\right)\left(f_L(x) - f(x)\right), f_{\partial,L}(x) - f_\partial(x)\right)^i\right)\right| \\
& && \text{(because } \psi(y) \in [0,1])\\
&\quad + 2D^2\alpha^B_{\max} \\
&\leq \gamma_\phi \mathit{\Gamma}^B_{\max} 2D^2 \alpha_{\max} + \frac{4dD\alpha_{\max}}{T} && \text{(by Lemma 9 and Proposition 10)}\\
&\quad + \gamma_\psi \mathit{\Gamma}^B_{\max} 2D^2 \alpha^B_{\max} && \text{(by Lemma 9 and since } \phi(y) \in [0,1])\\
&\quad + \left|\nabla\left(f_L(x) - f(x), f_{\partial,L}(x) - f_\partial(x)\right)^i\right| \\
&\quad + \left|\nabla\left(\left(\phi\left(|f_B|^2\right)\left(f_L(x) - f(x)\right), 0\right)^i\right)\right| && \text{(by the triangle inequality)}\\
&\quad + 2D^2\alpha^B_{\max} \\
&\leq \gamma_\phi \mathit{\Gamma}^B_{\max} 2D^2 \alpha^B_{\max} + \frac{4dD\alpha^B_{\max}}{T} && \text{(because by definition } \alpha_{\max} \leq \alpha^B_{\max})\\
&\quad + \gamma_\psi \mathit{\Gamma}^B_{\max} 2D^2 \alpha^B_{\max} \\
&\quad + \frac{4dD\alpha^B_{\max}}{T} && \text{(by Proposition 10)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \max_j \left|\nabla \left(\phi\left(|f_B|^2\right)\left(f_L^j(x) - f^j(x)\right)\right)\right| \\
&+ 2D^2\alpha_{\max}^B \\
&\leq (\mathit{\Gamma}_{\max}^B(\gamma_\phi + \gamma_\psi) + 1)2D^2\alpha_{\max}^B + \frac{8dD\alpha_{\max}^B}{T} \\
&+ \max_j \left|\nabla\phi\left(|f_B|^2\right)\right| \left|(f_L^j(x) - f^j(x))\right| \\
&+ \max_j \left|\phi\left(|f_B|^2\right)\right| \left|\nabla(f_L^j(x) - f^j(x))\right| \\
&\qquad\qquad\qquad\qquad \text{(By the Leibniz rule and the triangle inequality)} \\
&\leq (\mathit{\Gamma}_{\max}^B(\gamma_\phi + \gamma_\psi) + 1)2D^2\alpha_{\max}^B + \frac{8dD\alpha_{\max}^B}{T} \\
&+ \gamma_\phi \mathit{\Gamma}_{\max}^B 2D^2\alpha_{\max} \qquad \text{(by Lemma 27, (32), (33), and Lemma 9)} \\
&+ \frac{4dD\alpha_{\max}^B}{T} \qquad \text{(because } \phi(y) \in [0,1]\text{, Proposition 10, and } \alpha_{\max} \leq \alpha_{\max}^B\text{)} \\
&= (\mathit{\Gamma}_{\max}^B(2\gamma_\phi + \gamma_\psi) + 1)2D^2\alpha_{\max}^B + \frac{12dD\alpha_{\max}^B}{T} \stackrel{\text{def}}{=} e_t^{B_2} \qquad (51)
\end{aligned}
$$

We now write $\mathrm{G}^B = \mathrm{Gram}(\nabla f_B)$, $\widehat{\mathrm{G}}^{B_2} = \mathrm{Gram}(\nabla F_{L,2,\epsilon})$, and ${\mathrm{G}^B}_{i,j}$ and $\widehat{\mathrm{G}}_{i,j}^{B_2}$ for their (i,j)-th elements respectively. Similarly to (36) (we simply need to replace $F_{L,1}$ by $F_{L,2,\epsilon}$ and f by f_B), we obtain

$$
|\widehat{\mathrm{G}}_{i,j}^{B_2} - {\mathrm{G}^B}_{i,j}| \leq 2\gamma_{\max}^B e_t^{B_2} + (e_t^{B_2})^2 \qquad (52)
$$

By Corollary 8, we finally obtain

$$
|\hat{\lambda}_{\min}^{B_2} - \lambda_{\min}^B| \geq (d-n)\left(2\gamma_{\max}^B e_t^{B_2} + (e_t^{B_2})^2\right) \stackrel{\text{def}}{=} e_L^{B_2} \qquad (53)
$$

□

We again have the following corollary.

Corollary 41 ($F_{L,2,\epsilon}^{-1}(0)$ is a manifold) *We have that $F_{L,2,\epsilon}^{-1}(0)$ is a smooth manifold inside an small neighbourhood of $\sigma \times [0,1]$ provided that the following Regularity condition holds*

$$
\lambda_{\min}^B > e_L^{B_2}, \qquad (54)
$$

where $e_{PL}^{B_2}, = O(d^2 D)$ is precisely defined in (53).

3.2.3 Transversality with regard to the τ-direction

Once more similarly to Lemma 14 we have

Lemma 42 *Let $\mathit{\Xi}$ be as in Lemma 14. We have*

$$
\tan\angle(\nabla_{x,\tau}(F_{L,2,\epsilon}), \mathit{\Xi}) \leq \frac{2D^2\alpha_{\max}^B}{\sqrt{\lambda_{\min}^B} - e_t^{B_2}}
$$

where $e_t^{B_2} = O(dD)$ *is precisely defined in* (51)*. In particular, if the following Transversality condition holds:*

$$\sqrt{\lambda_{\min}^{B}} > e_t^{B_2} \tag{55}$$

the manifold $F_{L,2,\epsilon}^{-1}(0)$ *inside* $\sigma \times [0,1]$*, if well defined, is never tangent to the* $\tau = c$ *planes, where* c *is a constant.*

Proof The proof is identical to the proof of Lemma 14 with the replacement of $\alpha_{\max}$ by $\alpha_{\max}^{B}$ and of $\frac{4dD\alpha_{\max}}{T}$ in the denominator by $e_t^{B_2}$. The latter constant is a consequence of (51). □

Now, similary to Corollary 33, we have

Corollary 43 (Transversality with respect to τ for Step 2) *If both the Regularity condition* (54) *and the Transversality condition* (55) *hold, then, inside each* $\sigma \times [0,1]$*, the gradient of* τ *on* $F_{L,2,\epsilon}^{-1}(0)$ *is smooth and does not vanish. Both conditions hold if* $D = O(1/d^2)$*.*

3.2.4 Global result

We now have to prove that $F_{PL,2,\epsilon}^{-1}(0)$ is a manifold, for all sufficiently small ϵ. For this we first need the following bound, which is similar to the one in Lemma 34.

Lemma 44 *Let* v *be a vertex in* $\mathcal{T}$*,* $x_1, x_2 \in star(v)$*, and* $\tau_1, \tau_2 \in [0,1]$*, such that* $\nabla_{x,\tau} F_{PL,2,\epsilon}^{i}(x_1, \tau_1)$ *and* $\nabla_{x,\tau} F_{PL,2,\epsilon}^{i}(x_2, \tau_2)$ *are well defined, then*

$$|\nabla_{x,\tau} F_{PL,2,\epsilon}^{i}(x_1, \tau_1) - \nabla_{x,\tau} F_{PL,2,\epsilon}^{i}(x_2, \tau_2)| \leq g_{PL}^{B_2},$$

where $g_{PL}^{B_2} = \mathcal{O}(dD)$ *is precisely defined in* (56)*.*

Proof The proof follows the same steps as the proof of Lemma 34. By expansion we see that

$$\begin{aligned}
&|\nabla_{x,\tau} F_{PL,2,\epsilon}^{i}(x_1, \tau_1) - \nabla_{x,\tau} F_{PL,2,\epsilon}^{i}(x_2, \tau_2)| \\
&\quad = \big|\nabla_{x,\tau} F_{PL,2,\epsilon}^{i}(x_1, \tau_1) - \nabla_{x,\tau} f_B^{i}(x_1) \\
&\qquad + \nabla_{x,\tau} f_B^{i}(x_1) - \nabla_{x,\tau} f_B^{i}(x_2) \\
&\qquad + \nabla_{x,\tau} f_B^{i}(x_2) - \nabla_{x,\tau} F_{PL,2,\epsilon}^{i}(x_2, \tau_2)\big| \\
&\quad \leq \big|\nabla_{x,\tau} F_{PL,2,\epsilon}^{i}(x_1, \tau_1) - \nabla_{x,\tau} f_B^{i}(x_1)\big| \\
&\qquad + \big|\nabla_{x,\tau} f_B^{i}(x_1) - \nabla_{x,\tau} f_B^{i}(x_2)\big| \\
&\qquad + \big|\nabla_{x,\tau} f_B^{i}(x_2) - \nabla_{x,\tau} F_{PL,2,\epsilon}^{i}(x_2, \tau_2)\big| \qquad \text{(by the triangle inequality)} \\
&\quad \leq 2d\alpha_{\max}^{B} D + 2e_t^{B_2} \qquad \text{(by (8), } |x_1 - x_2| \leq 2D\text{, and (51) twice)} \\
&\quad \stackrel{\text{def}}{=} g_{PL}^{B_2}
\end{aligned} \tag{56}$$

□

Suppose $x_0, x_1, \ldots, x_m \in \mathrm{star}(v)$, $\tau_0, \ldots, \tau_m \in [0,1]$, and that, for all i, $\nabla_{x,\tau} F^i_{PL,2}(x_i, \tau_i)$ is well defined. Further assume that $\mu_1, \ldots, \mu_m$ are positive weights such that $\mu_1 + \cdots + \mu_m = 1$. We write

$$\widehat{\mathbf{G}}^{B_2} = \mathrm{Gram}\left(\sum_{k=1}^{m} \mu_k \nabla_{x,\tau} F_{PL,2}(x_k, \tau_k)\right), \tag{57}$$

and $\widehat{\mathbf{\Lambda}}^{B_2}_{\min}$ for the smallest modulus of the eigenvalues of $\widehat{\mathbf{G}}^{B_2}$.

Lemma 45

$$|\widehat{\mathbf{\Lambda}}^{B_2}_{\min} - \lambda^B_{\min}| \leq e^{B_2}_{PL} \tag{58}$$

where $e^{B_2}_{PL} = O(d^2 D)$ is precisely defined in (60).

Proof The proof is more or less the same as the proof of Lemma 35. Let $x_0 \in \mathrm{star}(v)$ and $\tau_0 \in [0,1]$, be such that $\nabla_{x,\tau} F^i_{PL,2,\epsilon}(x_0, \tau_0)$ is well defined. Note that it is sufficient for x_0 to lie in the interior of a d-simplex in $\mathcal{T}$. Lemma 40 gives that

$$|\widehat{\mathbf{\Lambda}}^{B_2}_{\min} - \lambda^B_{\min}| \leq e^{B_2}_{L}$$

Using $\nabla(f^i_B) \leq \gamma^B_{\max}$ and (51), we get

$$\left|\nabla_{x,\tau} F^i_{PL,2,\epsilon}(x_0, \tau_0)\right| \leq \gamma^B_{\max} + e^{B_2}_t \tag{59}$$

We want to bound the smallest absolute value of the eigenvalues of $\widehat{\mathrm{G}}^{B_2} = \mathrm{Gram}(\nabla_{x,\tau} F_{L,2,\epsilon}))$. We proceed similarly to (43) (with $F_{PL,1}$ replaced by $F^i_{PL,2,\epsilon}$). Let $\widehat{\mathbf{G}}^{B_2}$ be as in (57), and denote by $\widehat{\mathrm{G}}^{B_2}_{i,j}$ and $\widehat{\mathbf{G}}^{B_2}_{i,j}$ the (i,j) elements of $\widehat{\mathrm{G}}^{B_2}$ and $\widehat{\mathbf{G}}^{B_2}$ respectively, and by $\widehat{\mathbf{\Lambda}}^{B_2}_{\min}$ the smallest absolute value of the eigenvalues of $\widehat{\mathbf{G}}^{B_2}$.

$$|\widehat{\mathbf{G}}^{B_2}_{i,j} - \widehat{\mathrm{G}}^{B_2}_{i,j}| \leq \; 2\left(g^{B_2}_{PL}\right) \cdot (\gamma^B_{\max} + e^{B_2}_t + (g^{B_2}_{PL})^2) \qquad \text{(by (59))}$$

Thanks to Corollary 8 and Lemma 40, we have that

$$\begin{aligned} |\widehat{\mathbf{\Lambda}}^{B_2}_{\min} - \lambda^B_{\min}| &\leq e^{B_2}_L + (d-n)(2g^{B_2}_{PL}) \cdot \left(\gamma^B_{\max} + e^{B_2}_t + (g^{B_2}_{PL})^2\right) \\ &\stackrel{\mathrm{def}}{=} e^{B_2}_{PL} \end{aligned} \tag{60}$$

□

Lemma 45 immediately yields that

Corollary 46 (The generalized implicit function theorem in Step 2) *If the Regularity condition*

$$\lambda^B_{\min} > e^{B_2}_{PL}, \tag{61}$$

the generalized implicit function theorem, Theorem 19, applies to $F_{PL,1}(x,\tau) = 0$. In particular $F^{-1}_{PL,1}(0)$ is a manifold.

Theorem 47 *If the Regularity conditions* (39) *and* (61) *and the Transversality conditions* (40) *and* (55) *hold, then $f^{-1}(0) \cap f^{-1}_{\partial}([0,\infty))$ is isotopic to $f^{-1}_{PL}(0) \cap f^{-1}_{\partial,PL}([0,\infty))$. All conditions hold when choosing $D = O(1/d^2)$.*

Proof The proof follows from Corollaries 31, 33, 36, 41, 43, and 46. □

3.3 Fréchet distance

The bounds on the Fréchet distance can be achieved in the same way as before.

Theorem 48 *Suppose that the conditions of Theorem 47 are satisfied. Then*

$$d_F(f^{-1}(0), f_{PL}^{-1}(0)) \leq d_{PL}^B$$

where $d_{PL}^B = O(D^2)$ *is defined in* (62).

Proof We apply the same argument as in Lemma 25 and Corollary 26, for both steps of the proof. This yields the sum of two terms that are of the same form as (28). For the first step, we need the following substitutions:

- θ is replaced by $\theta_1 \stackrel{\text{def}}{=} \arctan \frac{2D^2\alpha_{\max}}{\sqrt{\lambda_{\min} - e_t^{B_1}}}$, as a consequence of Lemma 32.
- $\lambda_{\min} - e_{PL}$ is replaced by $\lambda_{\min} - e_{PL}^{B_1}$, as a consequence of Lemma 35.
- $\gamma_{\max} + \frac{4dD\alpha_{\max}}{T}$ is replaced by $\gamma_{\max} + 2D^2\alpha_{\max} + e_t^{B_1}$, as a consequence of (35).

For the second step we need the following substitutions:

- θ is replaced by $\theta_2 \stackrel{\text{def}}{=} \arctan \frac{2D^2\alpha^B_{\max}}{\sqrt{\lambda^B_{\min} - e_t^{B_2}}}$, as a consequence of Lemma 42.
- $\sqrt{\lambda_{\min} - e_{PL}}$ is replaced by $\sqrt{\lambda_{\min} - e_{PL}^{B_2}}$, as a consequence of Lemma 45.
- $\gamma_{\max} + \frac{4dD\alpha_{\max}}{T}$ is replaced by $\gamma^B_{\max} + e_t^{B_2}$ as a consequence of (51).

This yields

$$\begin{aligned} &d_F(f^{-1}(0) \cap f_\partial^{-1}([0,\infty)), f_{PL}^{-1}(0) \cap f_{\partial,PL}^{-1}([0,\infty))) \\ &\quad\leq \tan\arcsin \frac{\sin(\theta_1)(\gamma_{\max} + 2D^2\alpha_{\max} + e_t^{B_1})}{\sqrt{\lambda_{\min} - e_{PL}^{B_1}}} \\ &\qquad + \tan\arcsin \frac{\sin(\theta_2)(\gamma^B_{\max} + e_t^{B_2})}{\sqrt{\lambda_{\min} - e_{PL}^{B_2}}} \\ &\quad\stackrel{\text{def}}{=} d_{PL}^B. \end{aligned} \tag{62}$$

□

4 Isostratifolds

There is no obstruction that prevents us from extending the approach above to isostratifolds. By isostratifolds we mean stratifolds that are given by the zero sets of functions and inequalities. For example suppose that we want to find a PL approximation of the unit sphere centred at 0 in $\mathbb{R}^3$ including the PL approximations of the intersections of the sphere with slightly deformed

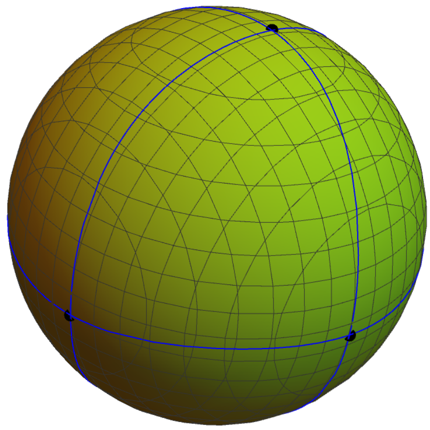

Fig. 3 An example of an isostratifold.

$x = 0$, $y = 0$, and $z = 0$-planes, as depicted in Figure 3. This would also give PL approximations of the respective 'octants' of the sphere.

We could follow the same procedure as for a manifold with boundary to give precise bounds on the longest edge length D of the ambient triangulation that ensure that the PL approximation is correct. However this would mean that we have to introduce an extra bump function for each stratum as well as an extra isotopy. Even though this should be relatively straightforward, finding the precise constants involved would become prohibitively lengthy.

5 Robustness

Suppose that f and f_δ are smooth functions and moreover f_δ is small in terms of the C^1-topology. Thanks to the implicit function theorem, we know that if 0 is a regular value of f, the zero set of f and the zero set of the slightly perturbed function $f+f_\delta$ are isotopic. We now give quantitative conditions that guarantee that $f^{-1}(0)$ and $(f+f_\delta)^{-1}(0)$ are ambient isotopic. Let $f, f_\delta : \mathbb{R}^d \to \mathbb{R}^{d-n}$, and write

$$\tilde{\alpha}_{\max} = \max_x \Big(\max\{\max_i |\nabla(f^i_\delta)(x)|, \max_i |f^i_\delta(x)|\Big)\} \tag{63}$$

$$\tilde{\lambda}_{\min} = \min_{x \in f^{-1}([-\tilde{\alpha}_{\max}, \tilde{\alpha}_{\max}]^{d-n})} \lambda_{\min}(\mathrm{Gram}(\nabla f)) \tag{64}$$

$$\tilde{\gamma}_{\max} = \max_{x \in \mathbb{R}^d} \max_i |\nabla(f^i)|. \tag{65}$$

Theorem 49 *If*

$$\tilde{\lambda}_{\min} > (d-n)(4\tilde{\gamma}_{\max}\tilde{\alpha}_{\max} + \tilde{\alpha}^2_{\max}).$$

and $\sqrt{\tilde{\lambda}_{\min}} > \tilde{\alpha}_{\max}$ *then* $f^{-1}(0)$ *and* $(f+f_\delta)^{-1}(0)$ *are ambient isotopic.*

Proof We first note that if $|f^i(x)| > \tilde{\alpha}_{\max}$ then $f(x) + \tau f_\delta(x) \neq 0$ for all $\tau \in [0,1]$, so we can restrict our attention to $f^{-1}([-\tilde{\alpha}_{\max}, \tilde{\alpha}_{\max}]^{d-n})$, conform (64). The proof is similar to the proof presented in the previous sections, but much simpler because here all functions are smooth. We start with the function $F(x,\tau) = f(x) + \tau f_\delta(x)$, where $\tau \in [0,1]$. We, again, first establish that the zero set of this function is a $(n+1)$-dimensional manifold. Secondly, we will see that the gradient of τ restricted to this $(n+1)$-dimensional manifold never vanishes. As we have seen in the previous sections, this suffices to establish the isotopy $F^{-1}(0)$ from $f^{-1}(0)$ to $(f+f_\delta)^{-1}(0)$, by Lemma 3.

As before, it suffices to prove that $\lambda_{\min}(\nabla_{x,\tau} F) > 0$ to establish that $F^{-1}(0)$ is a manifold. We write

$$\widehat{\mathrm{G}} = \mathrm{Gram}(\nabla_{x,\tau} F) \quad \text{and} \quad \mathrm{G} = \mathrm{Gram}(\nabla f)$$

We find that

$$\begin{aligned}
|\widehat{\mathrm{G}}_{i,j} - \mathrm{G}_{i,j}| &= |(\nabla f^i + \tau \nabla f^i_\delta, f^i_\delta) \cdot (\nabla f^j + \tau \nabla f^j_\delta, f^j_\delta) - \nabla(f^i) \cdot \nabla(f^j)| \\
&= |\tau(\nabla f^i \cdot \nabla f^j_\delta + \nabla f^i_\delta \cdot \nabla f^j) + f^i_\delta f^j_\delta| \\
&\leq |\tau||\nabla f^i||\nabla f^j_\delta| + |\tau||\nabla f^i_\delta||\nabla f^j)| + |f^i_\delta||f^j_\delta| \\
&\qquad \text{(by the triangle inequality and Cauchy-Schwarz)} \\
&\leq 2|\tau|\tilde{\gamma}_{\max}\tilde{\alpha}_{\max} + \tilde{\alpha}^2_{\max} \qquad \text{(by (63), (64) and (65))} \\
&\leq 4\tilde{\gamma}_{\max}\tilde{\alpha}_{\max} + \tilde{\alpha}^2_{\max} \qquad \text{(because } |\tau| \leq 1\text{)}
\end{aligned}$$

Corollary 8 implies that $F^{-1}(0)$ is a manifold if

$$\tilde{\lambda}_{\min} > (d-n)(4\tilde{\gamma}_{\max}\tilde{\alpha}_{\max} + \tilde{\alpha}^2_{\max}).$$

Lemma 13 further yields that $|\nabla f^i| \geq \sqrt{\tilde{\lambda}_{\min}}$ in $f^{-1}([-\tilde{\alpha}_{\max}, \tilde{\alpha}_{\max}]^{d-n})$. This means that the x component of $\nabla_{x,\tau}(F)$ doesn't vanish (again in $f^{-1}([-\tilde{\alpha}_{\max}, \tilde{\alpha}_{\max}]^{d-n})$) as long as

$$\sqrt{\tilde{\lambda}_{\min}} > \tilde{\alpha}_{\max}.$$

□

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A Notations and overview of constants

For notations, we followed the following rules:

1. Greek letters, except τ and ϵ, are for constants related to functions.
2. We use $\widehat{\cdot}$ for quantities related to PL functions.
3. Capital letters such as D, T are quantities related to the triangulation $\mathcal{T}$.
4. Bounds on gradients are denoted by g_x^y
5. Bounds on eigenvalues are denoted by e_x^y
6. For convenience, we write $\nabla_{x,\tau} f(x) = \begin{pmatrix} \nabla f^i(x) \\ 0 \end{pmatrix}$.

Overview of constants

We give an overview. We write $\mathcal{T}_0$ for the set of all $\sigma \in \mathcal{T}$, such that $(f^i)^{-1}(0) \cap \sigma \neq$ for all i. We write $\mathcal{T}_B$ for all $\sigma \in \mathcal{T}$ such that $(\sum_l (f^l)^2 + (f_\partial)^2)^{-1}([-2y_0, 2y_0]) \cap \sigma \neq \emptyset$. We write

$$f_B(x) = (f(x), f_\partial(x)) \tag{47}$$

$$\lambda_{\min} = \min_{x\in\mathcal{T}_0} \min_k |E_k(\nabla(f^i)\cdot\nabla(f^j))_{i,j}|, \tag{3}$$

$$\gamma_{\max} = \max_{x\in\mathcal{T}_0} \max_i |\nabla(f^i)| \tag{2}$$

$$\gamma^B_{\max} = \max_{x\in\mathcal{T}_0} \max_i |\nabla(f^i_B)| \tag{48}$$

$$\Gamma^B_{\max} = \max_{x\in\mathcal{T}_0} \left|\nabla\left(|f_B|^2\right)\right| = 2\max_{x\in\mathcal{T}_0} \left|\sum_l f^l \nabla f^l + f_\partial \nabla f_\partial\right| \tag{33}$$

$$\alpha_{\max} = \max_{x\in\mathcal{T}_0} \max_i \|\mathrm{Hes}(f^i)\|_2 = \max_x \max_i \|(\partial_k\partial_l f^i)_{k,l}\|_2 \tag{4}$$

$$\alpha^B_{\max} = \max_{x\in\mathcal{T}_0} \max_i \|\mathrm{Hes}(f^i_B)\|_2 = \max_{x\in\mathcal{T}_0} \max_i \|(\partial_k\partial_l f^i_B)_{k,l}\|_2 \tag{49}$$

D : the longest edge length of a simplex in $\mathcal{T}_0$

T : the smallest thickness of a simplex in $\mathcal{T}_0$.

$\varXi = \mathbb{R}^d \subset \mathbb{R}^{d+1}$ is the space spanned by the d basis vectors corresponding to the x-directions. The bump functions give rise to the following:

$$\gamma_\phi = 4\frac{e^{\frac{2}{3y_0}}}{y_0} \tag{32}$$

$$\gamma_\psi = \frac{200}{99}\frac{e^{\frac{400}{297y_0}}}{y_0}. \tag{50}$$

Bounds on gradients

$$g_{PL} = 2dD\alpha_{\max} + \frac{8dD\alpha_{\max}}{T} + 4D^2\alpha_{\max} \tag{21}$$

$$g_{PL}^{B_1} = 2d\alpha_{\max}D + 2\left((1+\gamma_{\max}^B\gamma_\phi)2D^2\alpha_{\max} + \frac{4dD\alpha_{\max}}{T}\right) \tag{41}$$

$$g_{PL}^{B_2} = 2d\alpha_{\max}^B D + 2e_t^{B_2} \tag{56}$$

Bounds on eigenvalues

$$e'_L = 2\gamma_{\max}\frac{4dD\alpha_{\max}}{T} + \left(\frac{4dD\alpha_{\max}}{T}\right)^2 \tag{15}$$

$$e_L = (d-n)(e'_L + (2D^2\alpha_{\max})^2) \tag{16}$$

$$e_{PL} = (d-n)g_{PL}(\gamma_1 + \frac{4dD\alpha_{\max}}{T} + 2D^2\alpha_{\max}) \tag{23}$$

$$e_t^{B_1} = 2D^2\alpha_{\max} + 2\gamma_{\max}^B\gamma_\phi D^2\alpha_{\max} + \frac{4dD\alpha_{\max}}{T} - 2D^2\alpha_{\max} \tag{35}$$

$$e_L^{B_1} = (d-n)2\gamma_{\max}\left(2D^2\alpha_{\max} + e_t^{B_1}\right) + \left(2D^2\alpha_{\max} + e_t^{B_1}\right)^2 \tag{38}$$

$$e_{PL}^{B_1} = e_L^{B_1} - (d-n)\left(2g_{PL}^{B_1}\cdot\left(\gamma_{\max} + 2D^2\alpha_{\max} + e_t^{B_1}\right) + \left(g_{PL}^{B_1}\right)^2\right) \tag{44}$$

$$e_t^{B_2} = (\gamma_{\max}^B(2\gamma_\phi + \gamma_\psi) + 1)2D^2\alpha_{\max}^B + \frac{12dD\alpha_{\max}^B}{T} \tag{51}$$

$$e_L^{B_2} = (d-n)\left(2\gamma_{\max}^B e_t^{B_2} + (e_t^{B_2})^2\right) \tag{53}$$

$$e_{PL}^{B_2} = e_L^{B_2} + (d-n)(2g_{PL}^{B_2})\cdot\left(\gamma_{\max}^B + e_t^{B_2} + (g_{PL}^{B_2})^2\right) \tag{60}$$

Bounds on angles

$$\theta = \frac{2D^2\alpha_{\max}}{\sqrt{\lambda_{\min}} - \frac{4dD\alpha_{\max}}{T}} \tag{18}$$